Automatica, Vol. 29, No. 3, pp. 661–679, 1993
Printed in Great Britain.

0005-1098/93 $6.00 + 0.00

Robust Filtering and Feedforward Control Based on Probabilistic Descriptions of Model Errors*

MIKAEL STERNAD† and ANDERS AHLÉN†

If model errors are represented by stochastic variables, performance robustness can be optimized by using a polynomial equations approach. Simple closed-form solutions minimize quadratic criteria, averaged with respect to model error distributions.

Key Words—Robustness; optimal estimation; Wiener filtering; deconvolution; prediction; feedforward control; model matching; smoothing; state estimation.

Abstract—A new approach to robust estimation of signals, prediction of time-series and robust feedforward control is considered. Modelling errors are parametrized by random variables, with known covariances. A robust design is obtained by minimizing the squared estimation error, averaged both with respect to model errors and the noise. A polynomial equations approach, based on averaged spectral factorizations and averaged Diophantine equations, is derived. Mild solvability conditions guarantee the existence of stable optimal filters and feedforward regulators. The robust design turns out to be no more complicated than the design of an ordinary Wiener filter or LQG regulator.

The proposed approach avoids two drawbacks of minimax design. First, probabilistic descriptions of model uncertainties may have soft bounds. These are more readily obtainable in a noisy environment than the hard bounds required for minimax design. Furthermore, not only the range of uncertainties, but also their likelihood is taken into account; common model deviations will have a greater impact on an estimator design than do very rare "worst cases". The conservativeness is thus reduced.

1. INTRODUCTION

In ROBUST FILTER SYNTHESIS, the ever present model uncertainty, and the whole range of expected system behaviour, is taken into account. We here propose a novel approach to robust design for open-loop problems such as signal estimation, state estimation and feedforward control. It is based on a stochastic description of model errors, shown to be related to the stochastic embedding concept of Goodwin and Salgado (1989). Compared to alternatives in the literature, the proposed method is simple and straightforward, yet flexible. It constitutes a generalization of the polynomial equations approach, which was pioneered by Kučera (1979, 1981, 1991).

In robust design problems, the true system is only partially known. Robust estimators are based on nominal models, $\mathscr{F}_0$, of transfer functions. Error models are also required. They specify modelling errors $\Delta\mathscr{F}$ as belonging to a certain class of systems. Nominal and error models will, taken together, be called extended design models. Error models are by necessity imprecise; exact modelling of the unmodelled dynamics would be a contradiction in terms. Hard bounds in the frequency domain are one example of specifications, commonly characterized as

$$|\Delta\mathscr{F}(\omega)| \le L(\omega). \tag{1.1}$$

For other models of spectral uncertainty, see Kassam and Poor (1985) or Goodwin and Salgado (1989). Error models may be obtained from off-line experiments or via on-line adaptation. Ways to utilize estimates of model uncertainty are of value in adaptive as well as in fixed filter design.

* Received 23 October 1991; revised 30 June 1992; received in final form 5 September 1992. The original version of this paper was presented at the 2nd IFAC Workshop on System Structure and Control which was held in Prague, Czechoslovakia during September 1992. The Published Proceedings of this IFAC Meeting may be ordered from: Pergamon Press Ltd, Headington Hill Hall, Oxford, OX3 0BW, U.K. This paper was recommended for publication in revised form by Associated Editor T. Başar under the direction of Editor H. Kwakernaak.

† Systems and Control Group, Uppsala University, P.O. Box 27 S-751 03 Uppsala, Sweden. The work has been partially supported by the Swedish Board for Technical Development, under contract 8701573 and by the Swedish Institute. It was carried out while Anders Ahlén was on leave at The Department of Electrical Engineering and Computer Science, The University of Newcastle, Australia.

We will consider mainly estimation problems, with scalar stationary measurements in discrete time. During the last two decades, several schemes for robust estimation have been suggested. The most obvious *ad hoc* approach is perhaps to detune a filter, by increasing the measurement noise variance used in the design. Optimization of H_∞-criteria has been applied to estimation problems. So far, however, the H_∞-criterion appears to be of limited utility for obtaining robust filters.† Instead, almost all systematic schemes have considered minimax optimization, usually of the mean square estimation error (MSE). This approach goes back to work by D'Appolito and Hutchinson (1972) and Leondes and Pearson (1972) who dealt with large uncertainties in the noise covariances as well as in the plant dynamics.

Minimax design is simplified if there exists a saddle-point solution. One may then search for a least favourable pair of signal and noise spectra, in prespecified uncertainty classes. Under certain conditions, the optimal estimator is a filter designed for that pair. See Kassam and Poor (1985) and also Kassam and Lim (1977), Poor (1980), Moustakides and Kassam (1983, 1985) and Vastola and Poor (1984). Uncertainties can be described in a state space framework. See e.g. Martin and Minz (1983), Wang *et al.* (1987) or Haddad and Bernstein (1988). The computational effort involved is considerable. Closed-form solutions mostly do not exist. Furthermore, minimax implies a worst case design, with the following drawbacks.

- Even when extremely unlikely, a worst case still determines the estimator completely. There is a considerable risk that this may lead to a very conservative design, with poor performance in the normal range of model errors. (The conservativeness can be reduced by using fictitious tighter model error bounds. But then the whole point of using a minimax approach becomes unclear. Why design for a worst case which is not really worst?)
- An error model with hard bounds is required. Otherwise, a worst case is mostly impossible to find. In noisy environments, it may be possible to obtain statistical 'soft' model error bounds, but much more difficult to obtain reliable hard bounds, such as (1.1).‡ Parameter errors obtained by system identification have, asymptotically, a Gaussian distribution under very general conditions. See e.g. Ljung (1987). With a distribution with infinite tails, hard bounds cannot really be justified.

There do exist applications where worst case design is well motivated. However, the mean performance of estimators and regulators is normally of greater interest. We will consider situations where one time-invariant linear filter is to be applied on a number of systems, with different dynamics. A probabilistic description of model errors is then appropriate. We utilize the criterion

$$\bar{E}(E\,|z(t)|^2), \tag{1.2}$$

where $z(t)$ is the estimation error, E denotes expectation over noise and $\bar{E}$ is the expectation over a (continuous or discrete) distribution of systems. If, for example, there is a single uncertain parameter, attaining three possible values h_1, h_2, h_3 with probabilities p_1, p_2, p_3, respectively (1.2) becomes

$$p_1 Ez(t)^2\,|_{h=h_1} + p_2 Ez(t)^2\,|_{h=h_2} + p_3 Ez(t)^2\,|_{h=h_3}.$$

The criterion (1.2) takes not only the effect, but also the likelihood of different modelling errors into account. It has been proposed, and used, by Chung and Bélanger (1976) and by Grimble (1984). Related is also the work by Speyer and Gustafson (1975) who used a conditional expectation. For discrete distributions, Nahi and Knobbe (1976) derived robust filters of very high order. For continuous distributions, errors in time constants had to be assumed small, because these solutions were based on series expansion. However, as will be exemplified in Section 4, robust design is then of limited interest. One might often just as well use a nominal filter for small model errors. A quest for simpler and more useful design methods has motivated our present work. Its main contributions are listed below.

- A special type of stochastic error model for transfer functions is introduced in Section 3. We will argue that it is flexible enough to describe a wide range of uncertainties, structured and unstructured, large and small,

† For methods, see e.g. Nagpal and Khargonekar (1991). The main known connection of H_∞-filtering to robustness, is the case when both signal and noise models have multiplicative errors, for which only power amplification bounds are known (Grimble and ElSayed, 1989). Then, an H_∞-optimal filter can be shown to minimize the worst case mean square error. However, such situations are of rather limited practical relevance. A less conservative design could possibly be obtained from a H_2/H_∞ approach. See Limebeer *et al.* (1991).

‡ If the noise is bounded, hard bounds may be obtained. See Fogel and Huang (1982). These bounds are extremely wide, compared to statistical standard deviation estimates. Furthermore if noise bounds are over-estimated, parameter error bounds do not converge to zero with an increasing number of data. If noise bounds are under-estimated, parameter set identification results in an empty set; algorithms such as those of Fogel and Huang will fail.

with soft or hard bounds. Besides series expansion, it may be obtained from consideration of time-domain responses, identification by functional series expansion or from a stochastic frequency-domain description. Under mild conditions, only second order moments of stochastic model coefficients need to be known.

- Based on this error model and on the criterion (1.2), a polynomial equations approach to H_2 (or Wiener) design for open-loop problems is developed in Section 4 and Appendix B. The design equations are of the same type as for the nominal solution: spectral factorizations and Diophantine equations. Only trivial additional computations are required for obtaining a robust design.
- In Sections 2–4, the method is developed in some detail for filtering, prediction and smoothing estimators of scalar signals, which we call cautious Wiener filters. The filters balance the importance of uncertainties due to noise and model errors. The performance difference between robust and nominal designs is also discussed. It can be extremely large when the nominal design is sensitive. It increases for large model errors and decreases with an increased noise level.
- In Section 5, a polynomial approach to robust state estimation is outlined.
- Section 6 presents design equations and a design example for robust disturbance measurement feedforward regulators and model matching or reference feedforward filters. The way in which uncertainties in different transfer functions affect the solution is explained.

Remarks on the notation. For any complex polynomial in the backward shift operator q^{-1}, of degree np, $P(q^{-1}) = p_0 + p_1 q^{-1} + \cdots + p_{np} q^{-np}$, the conjugate polynomial is defined as $P_*(q) \triangleq p_0^* + p_1^* q + \cdots + p_{np}^* q^{np}$. For matrices H, H^* means conjugate transpose. For polynomial matrices $\mathbf{P}(q^{-1})$, $\mathbf{P}_* \triangleq \mathbf{P}^*(q)$. In the frequency domain, z or $e^{i\omega}$ will be substituted for q. For convenience, polynomial arguments are often omitted. We call $P(q^{-1})$ stable if all zeros of $P(z^{-1})$ are in $|z| < 1$. Note that whenever P is stable, all zeros of P_* are in $|z| > 1$.

2. A SIGNAL ESTIMATION PROBLEM

2.1. *The problem set-up*

In the Sections 2–4, a generalized deconvolution problem will be considered, to illustrate the design principles. It includes, e.g. ordinary output filtering and prediction of ARMA-processes as special cases. It also includes the design of linear recursive equalizers for digital communications. Measurements are described as

$$y(t) = \mathscr{G}(q^{-1})u(t-k) + w(t). \qquad (2.1)$$

The stable, linear, causal, and possibly uncertain transfer function $\mathscr{G}(q^{-1})$ may e.g. represent a transducer or a transmission channel. The delay k denotes the minimum value of a possibly incompletely known time delay. The input $u(t)$ and the measurement noise $w(t)$ are described by possibly uncertain ARMA models

$$\begin{gathered} u(t) = \mathscr{F}(q^{-1})e(t); \quad w(t) = \mathscr{H}(q^{-1})v(t) \\ E\,|e(t)|^2 = 1; \quad E\,|v(t)|^2 \triangleq \rho. \end{gathered} \qquad (2.2)$$

The white time-series $e(t)$ and $v(t)$ are assumed mutually uncorrelated. They are stationary, with zero mean. All transfer functions are assumed time-invariant. In the next section, they will be partitioned into exactly known nominal models and additive errors $\mathscr{F}(q^{-1}) = \mathscr{F}_0(q^{-1}) + \Delta\mathscr{F}(q^{-1})$, etc. as in Fig. 1. For now, we represent the extended design models as fractions of (partly known) polynomials,

$$\begin{aligned} \mathscr{F}(q^{-1}) &\triangleq \frac{C(q^{-1})}{D(q^{-1})}, \\ \mathscr{G}(q^{-1}) &\triangleq \frac{B(q^{-1})}{A(q^{-1})}, \\ \mathscr{H}(q^{-1}) &\triangleq \frac{M(q^{-1})}{N(q^{-1})}. \end{aligned} \qquad (2.3)$$

The polynomial degrees nc, nd, etc. are known to (or determined by) the designer.† The polynomials D, A and N in (2.3) are all assumed stable. Model uncertainty more or less forces us to restrict attention to sets of stable systems.‡ Signals and polynomial coefficients may be

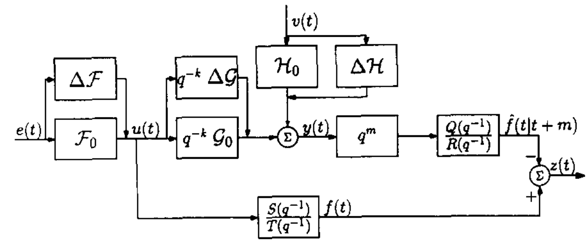


FIG. 1. A general scalar filtering problem. A filtered version $f(t)$ of the signal $u(t)$ is to be estimated, from noisy measurements $y(t)$. The uncertainties in transfer functions are represented by additive model errors $\Delta\mathscr{F}$, $\Delta\mathscr{G}$ and $\Delta\mathscr{H}$, described by stochastic error models.

† Note that we are talking about extended design models. In practice, they might only be approximations of classes of possibly infinite dimensional and time-varying true systems.

‡ With uncertain unstable poles, the design problem becomes unsolvable, in the open-loop context considered here. Thus, only a single Diophantine equation will be required. (If unstable poles were exactly known, a finite estimation error could, theoretically, be obtained.)

complex-valued; this is the case, e.g. in digital communications applications. A stable, linear and time-invariant estimator

$$\hat{f}(t \mid t+m) = \frac{Q(q^{-1})}{R(q^{-1})} y(t+m), \quad (2.4)$$

of the possibly filtered input

$$f(t) = \frac{S(q^{-1})}{T(q^{-1})} u(t), \quad (2.5)$$

is sought. See Fig. 1. The stable filter S/T is assumed to be specified by the user, and thus to be exactly known.† Depending on m, the estimator is a predictor ($m<0$), a filter ($m=0$) or a fixed lag smoother ($m>0$). We minimize (1.2), or

$$\bar{E}(E\,|z(t)|^2) = \bar{E}E\,|f(t) - \hat{f}(t \mid t+m)|^2. \quad (2.6)$$

2.2. *The nominal solution*

Assume a stable nominal model $\{C_0/D_0, B_0/A_0, M_0/N_0\}$ of (2.1)–(2.3) to be known. Assume C_0B_0 and ρM_0 to have no common zeros on $|z|=1$. (When $\rho=0$, C_0 and B_0 are not allowed to have zeros on $|z|=1$.) A stable estimator may then be designed to minimize the mean square error (MSE) $E\,|f(t)-\hat{f}(t \mid t+m)|^2$, without taking the model uncertainty into account. This estimator is given by

$$\hat{f}_0(t \mid t+m) = \frac{Q_{10}N_0A_0}{T\beta_0} y(t+m). \quad (2.7)$$

Here, $\beta_0(q^{-1})$ is the numerator of the nominal innovations model of $y(t)$. It is the stable and monic solution of the polynomial spectral factorization equation

$$r_0\beta_0\beta_{0*} = C_0C_{0*}B_0B_{0*}N_0N_{0*} + \rho M_0M_{0*}A_0A_{0*}D_0D_{0*}, \quad (2.8)$$

with r_0 being a scalar. The polynomial $Q_{10}(q^{-1})$, together with a polynomial $L_{0*}(q)$, is the unique solution to the Diophantine equation

$$q^{-m+k}SC_0C_{0*}B_{0*}N_{0*} = r_0\beta_{0*}Q_{10} + qD_0TL_{0*}. \quad (2.9)$$

If the true system equals the nominal model, the minimal MSE is

$$E\,|z(t)|_0^2 = \frac{1}{2\pi i}\oint \frac{L_0L_{0*}}{r_0\beta_0\beta_{0*}} + \rho\frac{SS_*}{TT_*} \times \frac{C_0C_{0*}M_0M_{0*}A_0A_{0*}}{r_0\beta_0\beta_{0*}}\frac{dz}{z}. \quad (2.10)$$

This is a Wiener filter, optimized by using polynomial equations. Its derivation is a special case of that in Appendix B. See Ahlén and Sternad (1989) for details.

3. PROBABILISTIC ERROR MODELS

3.1. *Properties of the error models*

Our goal is now to obtain a simple closed-form solution to the robust estimation problem (2.1)–(2.6). The perhaps most obvious way of parametrizing errors in polynomial coefficients,

$$\frac{C}{D} = \frac{C_0 + \Delta C}{D_0 + \Delta D},$$

is unsuitable in this respect. The reason is that ΔD would not, as opposed to ΔC, appear linearly when expectations with respect to parameters are evaluated in the criterion. This would cause severe difficulties in the derivation of design equations. A crucial simplification is to proceed instead from error models which are linear in all parameters treated as stochastic variables. Additive transfer functions $\Delta\mathcal{F}$, $\Delta\mathcal{G}$, $\Delta\mathcal{H}$, with partly unknown stochastic numerators and pre-specified denominators, will be considered. They provide flexibility, without sacrificing the linearity mentioned above. The extended design model (2.1)–(2.3) is thus specified as

$$\begin{aligned} \frac{C}{D} &= \frac{C_0}{D_0} + \frac{C_1\,\Delta C}{D_1} = \frac{C_0D_1 + D_0C_1\,\Delta C}{D_0D_1}, \\ \frac{B}{A} &= \frac{B_0}{A_0} + \frac{B_1\,\Delta B}{A_1} = \frac{B_0A_1 + A_0B_1\,\Delta B}{A_0A_1}, \\ \frac{M}{N} &= \frac{M_0}{N_0} + \frac{M_1\,\Delta M}{N_1} = \frac{M_0N_1 + N_0M_1\,\Delta M}{N_0N_1}. \end{aligned} \quad (3.1)$$

Above, the nominal models C_0/D_0, etc. are assumed known and stable, with degrees nc_0, nd_0, etc. Stable "error denominators" D_1, A_1 and N_1, of degrees nd_1, na_1 and nn_1, as well as numerator factors C_1, B_1, M_1, of degrees nc_1, nb_1 and nm_1, may be specified by the designer or obtained from data. Coefficients of numerator polynomials

$$\Delta P(q^{-1}) = \Delta p_0 + \Delta p_1 q^{-1} + \cdots + \Delta p_{\delta_p} q^{-\delta p}, \quad (3.2)$$

are stochastic variables. They have zero means and parameter covariances $\bar{E}\,\Delta p_i\,\Delta p_j^*$, collected

† It can be used, e.g. to reduce the estimator gain outside a restricted interesting frequency range, see Ahlén and Sternad (1989). For another application, see Carlsson *et al.* (1991, 1992). Alternatively, one could use a filter in the criterion, $J = \bar{E}E\,|(S/T)(u(t) - \hat{u}(t \mid t+m))|^2$. The design equations to be discussed are easily modified to apply to that problem.

in covariance matrices $\mathbf{P}_{\Delta P}$. (The zero-mean property defines the nominal model.) These coefficients are constant in time, so they are independent of the time series $e(t)$ and $v(t)$. Except for first and second order moments, their distributions need not be known, since they will not affect the design. Decomposition of numerators $P_1 \Delta P$ into known factors P_1 and stochastic factors ΔP simplifies the uncertainty modelling. In the sequel, we utilize two mild assumptions.

(A1) The coefficients of ΔC are independent of those of ΔB.

(A2) The matrices $\mathbf{P}_{\Delta C}$, $\mathbf{P}_{\Delta B}$ and $\mathbf{P}_{\Delta M}$, containing covariances of the coefficients of $\Delta C, \Delta B$ and ΔM, are Hermitian and positive semidefinite.

It is necessary to assure (A2) when the covariance matrices are used pragmatically, as "robustness tuning knobs". Design equations may readily be derived for situations with correlations between ΔC and ΔB. Assumption (A1) does, however, simplify the solution, and seems reasonable.

Model error covariances may be obtained from identification experiments, or from frequency domain data on system variability. If a fixed filter is to be designed for a large number of systems, the statistics may be obtained from a representative sample of systems. Probabilistic error models remain useful also when statistics is hard to obtain. Those who prefer a Bayesian view could then interprete error distributions as subjective probabilities. Others may just use them pragmatically, as robustness "tuning knobs". The covariances are then altered until satisfactory spectral properties of the filter are obtained. Let us now illustrate the versatility of the structure (3.1), and outline principles for tuning it.

If only the signal to noise ratio is uncertain, we set $\Delta C = \Delta B = \Delta M = 0$, and use a higher equivalent noise variance. A model (3.1) with uncertain noise variance, but well-defined noise spectrum is given by $M/N = M_0/N_0 + M_0\,\Delta m_0/N_0 = (M_0/N_0)(1 + \Delta m_0)$, with a scalar stochastic Δm_0. It corresponds to regarding the noise as having variance $\rho(1 + \bar{E}\,|\Delta m_0|^2)$.

Another special case is the use of FIR- or MA-filters (i.e. no denominators):

$$u(t) = (C_0 + \Delta C)e(t),$$
$$y(t) = q^{-k}(B_0 + \Delta B)u(t) + (M_0 + \Delta M)v(t). \tag{3.3}$$

In (3.3), degrees of stochastic polynomials may be set higher than those of the nominal polynomials, $\delta c > nc_0$, etc. This can be used to guard against under-parametrization. However, for systems with long or infinite impulse responses, error models with denominators are more appropriate than FIR-models.

The structure (3.1) covers multiplicative as well as additive descriptions of model errors. A multiplicative error is obtained with, e.g. $B_1 = B_0 B_m$, $A_1 = A_0 A_m$, with B_m, A_m arbitrary. It can be useful to extend (3.1) with several additive model error terms. For example, the total error of an identified model can be described by a variance part and a bias part. When the model error is estimated from data, using the algorithm described in Goodwin *et al.* (1992), the result is a sum of two expressions of type

$$\bar{E}(\Delta\mathcal{G}_i(q^{-1})\Delta\mathcal{G}_{i*}(q)) = \frac{1}{A_{i1}(q^{-1})A_{i1*}(q)}(B_{i1}(q^{-1}) \cdots B_{in}(q^{-1}))P_i\begin{pmatrix} B_{i1*}(q) \\ \vdots \\ B_{in*}(q)\end{pmatrix},$$

which describe the bias ($i = 1$) and the variance ($i = 2$) contribution to the total error. Above, P_i are positive semidefinite matrices. (See (42) in Goodwin *et al.* (1992).) Such expressions can always be substituted by scalar models for the bias and variance, respectively. They can also be multiplied together, to obtain one scalar expression for the second order properties

$$\frac{G(q, q^{-1})}{A_1(q^{-1})A_{1*}(q)}.$$

In our framework, this corresponds to a single scalar error model $B_1\,\Delta B/A_1$ with $B_1 B_{1*}\bar{E}(\Delta B\,\Delta B_*) = G$. We can thus, without restrictions, confine the discussion to models with single error terms.

3.2. *Time-domain determination of error models*

Example 1: *an uncertain time constant.* Assume that a signal $u(t)$ is known to be generated by a low order system, with a white input $e(t)$. Its impulse response is known to start at 1.0, but then becomes uncertain, because the dominant pole location is uncertain. With 95% probability, the pole is believed to be in the range [0.75 . . . 0.95], with an average around 0.9. From a first order model of this signal

$$u(t) = \mathcal{F}e(t) = \frac{1}{1 - pq^{-1}}e(t), \tag{3.4}$$

we might choose the nominal model $u(t) =$

$\mathcal{F}_0 e(t)$, with

$$\mathcal{F}_0 = \frac{C_0}{D_0} = \frac{1}{1 - 0.9q^{-1}}. \quad (3.5)$$

An additive error model could be derived by a first order series expansion of $\mathcal{F}$ in the uncertain coefficient p

$$\Delta \mathcal{F} = \left(\frac{\partial \mathcal{F}}{\partial p} \bigg|_{p=0.9} \right) \Delta p = \frac{q^{-1} \Delta p}{(1 - 0.9q^{-1})^2}. \quad (3.6)$$

Here, Δp is seen as a zero mean stochastic variable. However, the fit of the impulse responses of $\mathcal{F} = \mathcal{F}_0 + \Delta \mathcal{F}$ to the expected range of impulse responses of the system (3.4) is not good, see Fig. 2. They tend to spread widely in the beginning and are too narrow for large t. A better approach is to keep the structure of (3.6), but with free user choices of denominator coefficients. Thus, we consider the error model

$$\Delta \mathcal{F} = \frac{C_1 \Delta C}{D_1} = \frac{q^{-1} \Delta c}{1 + d_1^1 q^{-1} + d_2^1 q^{-2}},$$

with Δc being a scalar stochastic variable. Its variance, and the coefficients of D_1, should be tailored to the actual amount of uncertainty. Better centering of the nominal model will also improve the fit. A solution which exactly fits the $\pm 95\%$-bounds can be found in this example. It is given by

$$\mathcal{F} = \frac{1 - 0.85q^{-1}}{(1 - 0.95q^{-1})(1 - 0.75q^{-1})} + \frac{q^{-1} \Delta c}{(1 - 0.95q^{-1})(1 - 0.75q^{-1})}, \quad (3.7)$$

with $\bar{E}(\Delta c)^2 = 0.0025$, if Δc is assumed Gaussian. See Fig. 2. The model (3.7) reduces to $\mathcal{F} = 1/(1 - 0.95q^{-1})$ when $\Delta c = 0.10$ (2σ-limit)

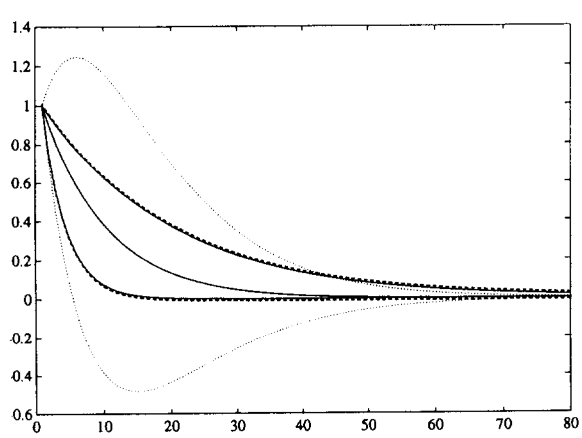


FIG. 2. Impulse responses of (3.4) in Example 1, for $p = 0.75$, 0.9 and 0.95 (solid). The $\pm 95\%$ probability limits of the extended design model $\mathcal{F} = \mathcal{F}_0 + \Delta \mathcal{F}$ are indicated for the model (3.7) (dashed) as well as for a series-expansion based Gaussian model (3.5), (3.6), with $\bar{E}(\Delta p)^2 = 0.010$ (dotted).

and to $\mathcal{F} = 1/(1 - 0.75q^{-1})$ when $\Delta c = -0.10$. A systematic algorithm for fitting error models to time-domain data is a topic of current research.

For a system with measurable inputs, another way of obtaining extended design models of the type (3.1) is from identification experiments based on functional series expansions $\sum_{i=1}^{M} p_i \mathcal{B}_i(q^{-1})$. Here, $\mathcal{B}_i$, $i = 1 \cdots M$ represent a set of predetermined rational basis functions, such as, e.g. discrete Laguerre functions. This model structure is linear in the parameters $\{p_i\}$. It has received increasing interest as a tool for system identification, see Wahlberg (1991) or Goodwin *et al.* (1991). If an identification experiment provides parameter estimates $\{\bar{p}_{0i}\}$ and covariances for zero mean errors $\{\Delta \bar{p}_i\}$, we obtain the extended design model

$$\bar{\mathcal{P}} \triangleq \sum_{i=1}^{M} \bar{p}_{0i} \mathcal{B}_i + \sum_{i=1}^{M} \Delta \bar{p}_i \mathcal{B}_i = \bar{\mathcal{P}}_0 + \Delta \bar{\mathcal{P}}.$$

If bias-errors are small, the covariance matrix provides acceptable estimates of the modelling errors. Writing $\Delta \bar{\mathcal{P}}$ in common denominator form and use of the covariance matrix for $\{\Delta \bar{p}_i\}$ gives $\bar{E}(\Delta \bar{\mathcal{P}} \Delta \bar{\mathcal{P}}_*)$, needed in the robust design.

3.3. *Frequency domain specifications*

Probabilistic error models can be specified in the frequency domain, as parametrized distributions. If the input is measurable, the parameters of such models can be estimated from data, using a maximum likelihood method. See Goodwin *et al.* (1992). Let us briefly recapitulate the stochastic embedding concept, proposed by Goodwin and Salgado (1989). An additive transfer function error $\Delta \mathcal{G}(e^{i\omega})$ is viewed as a realization of a stochastic process in the frequency domain, with zero mean and

$$\bar{E}\{\Delta \mathcal{G}(e^{i\omega_1}) \Delta \mathcal{G}_*(e^{i\omega_2})\} \triangleq \Gamma(e^{i\omega_1}, e^{i\omega_2}) \geq 0.$$

For stationary processes, $\Gamma(e^{i\omega_1}, e^{i\omega_2}) = \Gamma_s(e^{i(\omega_1 - \omega_2)})$. The shape of Γ_s is a measure of the assumed frequency domain smoothness of particular realizations of the model error. The variance ($\omega_1 - \omega_2 = 0$) is a scale factor for the uncertainty.

The frequency domain stochastic process $\Delta \mathcal{G}(e^{i\omega})$ corresponds to a time-domain filter with stochastic, zero mean, impulse response coefficients

$$\Delta \mathcal{G}(q^{-1}) = \sum_{j=0}^{\infty} g_j q^{-j}; \quad \bar{E}(g_j, g_\ell) = \gamma(j, \ell). \quad (3.8)$$

Here, $\gamma(j, \ell)$ can be calculated from the inverse two-dimensional discrete Fourier transform of $\Gamma(e^{i\omega_1}, e^{i\omega_2})$. For stationary stochastic processes in the frequency domain, the corresponding time-domain stochastic process will be nonstationary and white, with

$$\bar{E}(g_j, g_\ell) = \gamma_j\,\delta_{j,\ell}. \qquad (3.9)$$

For example, consider a frequency domain stochastic process $\mathscr{H}(e^{i\omega})$, with a zero mean Gaussian distribution and covariance function

$$\bar{E}\{\mathscr{H}(e^{i\omega_1})\mathscr{H}_*(e^{i\omega_2})\} = \frac{\alpha e^{i(\omega_1-\omega_2)}}{e^{i(\omega_1-\omega_2)} - \lambda}. \qquad (3.10)$$

It corresponds to the nonstationary time domain model (3.8), (3.9), with a Gaussian distribution of independent parameters, and variances $\gamma_j = \alpha\lambda^j$. See Goodwin *et al.* (1992). By truncating at some $j = M$ for which λ^M is small, we obtain $\mathscr{H}(q^{-1}) \approx h_0 + \cdots + h_M q^{-M}$, with $\bar{E}(h_j)^2 = \alpha\lambda^j$.

A priori information may be available about the frequency domain distribution of the unmodelled dynamics. It can be incorporated by using

$$\Delta\mathscr{G}(q^{-1}) = \mathscr{M}(q^{-1})\mathscr{N}'_\Delta(q^{-1}). \qquad (3.11)$$

Here, $\mathscr{M}(q^{-1})$ is a known frequency shaping filter, and $\mathscr{N}'_\Delta(q^{-1})$ is a stationary process in the frequency domain, with covariance function $\Gamma'_s(e^{i(\omega_1-\omega_2)})$.

Example 2: *a frequency-shaped error model.* Suppose that the mean magnitude of the model error in the frequency domain can be described by a magnitude response of a first order filter. Then, we may use (3.11), with

$$\mathscr{M}(q^{-1}) = \frac{1 + \eta q^{-1}}{1 + a_{11}q^{-1}}. \qquad (3.12)$$

Also, assume that the parameter λ in (3.10) can be tuned to give a reasonable description of the degree of smoothness (in the frequency domain) of the most probable model errors. The process in (3.10) can then be used to represent the stationary part $\Gamma'_s(e^{i(\omega_1-\omega_2)})$ of the frequency-domain process. Truncation of its corresponding time-domain impulse response gives a model (3.11), with the structure (3.1)

$$\Delta\mathscr{G}(q^{-1}) = \frac{1 + \eta q^{-1}}{1 + a_{11}q^{-1}}(h_0 + h_1 q^{-1} + \cdots + h_M q^{-M})$$
$$= \frac{B_1(q^{-1})\,\Delta B(q^{-1})}{A_1(q^{-1})}. \qquad (3.13)$$

The covariance matrix of $\{h_j\}$ is $\mathbf{P}_{\Delta B} = \operatorname{diag}(\alpha\lambda^j)$. Note that this model is fully determined by only five parameters, α, λ, a_{11}, η and the truncation length M.

4. DESIGN OF ROBUST FILTERS

We proceed from the model (2.1), (2.2), (2.3) and (3.1). The coefficients of ΔC, ΔB and ΔM are random variables. They cause the polynomials C, B and M to be random variables as well. The expectation operator $\bar{E}$ in (2.6) will provide an ensemble average, yielding one estimator to be used on all systems.

4.1. *The averaged spectral factorization*

An averaged spectral factor $\beta(q^{-1})$ is defined as the numerator polynomial of an averaged innovations model of the utilized measurement. In the present case, it is given by the stable and monic solution to

$$r\beta\beta_* \triangleq \bar{E}\{CC_*BB_*NN_* + \rho MM_*AA_*DD_*\}, \qquad (4.1)$$

with scalar r. Define double-sided polynomials

$$\tilde{C}\tilde{C}_* \triangleq \bar{E}(CC_*),$$
$$\tilde{B}\tilde{B}_* \triangleq \bar{E}(BB_*),$$
$$\tilde{M}\tilde{M}_* \triangleq \bar{E}(MM_*).$$

Then, use of (3.1) gives

$$\tilde{C}\tilde{C}_* = C_0C_{0*}D_1D_{1*} + D_0D_{0*}C_1C_{1*}\bar{E}(\Delta C\,\Delta C_*),$$
$$\tilde{B}\tilde{B}_* = B_0B_{0*}A_1A_{1*} + A_0A_{0*}B_1B_{1*}\bar{E}(\Delta B\,\Delta B_*),$$
$$\tilde{M}\tilde{M}_* = M_0M_{0*}N_1N_{1*} + N_0N_{0*}M_1M_{1*}\bar{E}(\Delta M\,\Delta M_*). \qquad (4.2)$$

We can now simplify (4.1).

Lemma 1. Let Assumption (A1) hold. Then, (4.1) can be expressed as

$$r\beta\beta_* = \tilde{C}\tilde{C}_*\tilde{B}\tilde{B}_*NN_* + \rho\tilde{M}\tilde{M}_*AA_*DD_*. \qquad (4.3)$$

Proof. The coefficients of a polynomial ΔP are zero mean stochastic variables. Therefore, the coefficients of $\Delta P\,\Delta P_*$ will also be stochastic variables, having expected values given by (4.5) below. The coefficients of $(\Delta B, \Delta C)$, are independent, and so are the coefficients of $\Delta B\,\Delta B_*$, $\Delta C\,\Delta C_*$. Using independence for complex parameters (see e.g. Loève (1963)), the right-hand side of (4.1) becomes

$$\bar{E}(CC_*)\bar{E}(BB_*)NN_* + \rho\bar{E}(MM_*)AA_*DD_*.$$

The averaged factors in (4.2) can be evaluated as follows. For a stochastic error model numerator $\Delta P(q^{-1})$, as in (3.2), let the Hermitian parameter covariance matrix be

$$\mathbf{P}_{\Delta P} = \begin{bmatrix} \bar{E}\,|\Delta p_0|^2 & \cdots & \bar{E}(\Delta p_0\,\Delta p^*_{\delta p}) \\ \vdots & \ddots & \vdots \\ \bar{E}(\Delta p_{\delta p}\,\Delta p^*_0) & \cdots & \bar{E}\,|\Delta p_{\delta p}|^2 \end{bmatrix}. \qquad (4.4)$$

Denote the sum of the diagonal elements g_0, the sum of the elements in the ith super-diagonal g_i, the sum of elements in the ith sub-diagonal g_{-i}. Note that $g_{-i} = g_i^*$. Then it becomes evident, by direct multiplication of $\Delta P(q^{-1})\,\Delta P_*(q)$, and taking expectations, that

$$\bar{E}(\Delta P\,\Delta P_*) = g_{dp}^* q^{-dp} + \cdots + g_1^* q^{-1} + g_0 + g_1 q + \cdots + g_{dp} q^{dp}. \quad (4.5)$$

Thus, the averaged factors in (4.2) are readily obtained. Above, $dp \le \delta p$, with $dp = 0$ if the coefficients are uncorrelated. For example, the coefficients h_i are uncorrelated in the model (3.13). The resulting polynomial $\bar{E}(\Delta B\,\Delta B_*)$ would then simply be a scalar, regardless of the degree of ΔB.

In (4.2), $\tilde{C}\tilde{C}_*$ will contain powers up to $q^{\pm n\tilde{c}}$, where $n\tilde{c} = \max\{nc_0 + nd_1,\ nd_0 + nc_1 + dc\}$, with analogous expressions for $n\tilde{b}, n\tilde{m}$. Since $N = N_0 N_1$, etc. the averaged spectral factor in (4.3) has degree

$$n\beta = \max\{n\tilde{c} + n\tilde{b} + nn_0 + nn_1,\ n\tilde{m} + na_0 + na_1 + nd_0 + nd_1\}.$$

The factorization (4.3) is solvable with respect to a unique stable $\beta(z^{-1})$ if and only if its right-hand side is positive on $|z| = 1$. Introduce the assumptions

(A3) C_0, $C_1\bar{E}(\Delta C\,\Delta C_*)$, ρM_0 and $\rho M_1\bar{E}(\Delta M\,\Delta M_*)$ have no common zeros on $|z| = 1$.

(A4) B_0, $B_1\bar{E}(\Delta B\,\Delta B_*)$, ρM_0 and $\rho M_1\bar{E}(\Delta M\,\Delta M_*)$ have no common zeros on $|z| = 1$.

Lemma 2. Let D, A and N be stable and (A2) hold. Then, a unique stable spectral factor β, satisfying (4.3), exists, if and only if both of (A3) and (A4) are true.

Proof. See Appendix A.

The conditions (A3) and (A4) are mild. They will almost always be fulfilled, even if C_0, B_0 and M_0 have zeros on the unit circle. In fact, the conditions are more relaxed than for the nominal case, due to the presence of averaged factors $\bar{E}(\cdot)$.

4.2. *The cautious Wiener filter*

Theorem 1. Assume an extended design model (2.1), (2.2), (2.3) and (3.1) to be given, with known covariances of the stochastic polynomial coefficients. Assume (A1)–(A4) to hold. An estimator of $f(t)$ then minimizes (2.6), among all linear time-invariant estimators based on $y(t+m)$, if and only if it has the same coprime factors as

$$\hat{f}(t \mid t+m) = \frac{Q_1 N_0 N_1 A_0 A_1}{T\beta}\, y(t+m). \quad (4.6)$$

Here, $\beta(q^{-1})$ is obtained from (4.3), while $Q_1(q^{-1})$, together with $L_*(q)$, is the unique solution to

$$q^{-m+k} S\tilde{C}\tilde{C}_* B_{0*} A_{1*} N_{0*} N_{1*} = r\beta_* Q_1 + q D_0 D_1 T L_*, \quad (4.7)$$

with generic† polynomial degrees

$$nQ_1 = \max(ns + n\tilde{c} - k + m,\ nd_0 + nd_1 + nt - 1),$$
$$nL = \max(n\tilde{c} + nb_0 + na_1 + nn_0 + nn_1 + k - m,\ n\beta) - 1. \quad (4.8)$$

For the ensemble of systems, the minimal criterion value becomes

$$\bar{E}E\,|z(t)|^2_{\min} = \frac{1}{2\pi i}\oint \frac{LL_*}{r\beta\beta_*} + \rho\frac{SS_*\tilde{C}\tilde{C}_*\tilde{M}\tilde{M}_* AA_*}{TT_* r\beta\beta_*} + \frac{SS_*\tilde{C}\tilde{C}_*\tilde{C}\tilde{C}_*\bar{E}(\Delta\mathscr{G}\,\Delta\mathscr{G}_*)AA_*NN_*}{TT_* DD_* r\beta\beta_*}\frac{dz}{z}. \quad (4.9)$$

Proof. See Appendix B.

Remarks. For minimizing (2.6), the design equations are (4.3), (4.5) and (4.7). The only new type of computation, as compared to a nominal solution, is trivial: summation of covariance matrix elements, diagonalwise. There will, of course, exist a filter giving superior MSE for any particular system, namely the (unknown) Wiener filter for that system. There may also exist superior nonlinear filters.

Note that N_1 and A_1 affect the filter (4.6) directly. If $1/N_1$ or $1/A_1$ in the error models have resonance peaks, indicating large uncertainty, the filter (4.6) will have low gain at those frequencies. Stable common factors might exist in (4.6). (Thus, the formulation about coprime factors.) Equation (4.7) will have a unique solution, with degrees (4.8). Note that $\beta_*(z)$ (unstable) and $D_0(z^{-1})D_1(z^{-1})T(z^{-1})$ (stable) have no common factors. Almost common factors close to $|z| = 1$ tend to make Diophantine equations numerically sensitive. The risk for this is less in (4.7) than in (2.9), because zeros of β are in general more distant from unit circle than zeros of β_0. The filter (4.6) tends to have lower resonance peaks than (2.7).

It is easy to show that if the block S/T would contain an uncertainty, $S/T = S_0/T_0 + S_1\,\Delta S/T_1$, this would not affect the resulting filter. In (4.6) and (4.7), the nominal polynomials S_0 and T_0 would be substituted for S and T.

† For special polynomials, the solution may have lower degrees.

If the variance of broad-band measurement noise is increased, the gains of both the nominal and the robust filters decrease. If the noise level is high, performance differences between nominal and robust solutions tend to be small.

It is shown in Appendix B that if no uncertainty is assumed, (4.3), (4.6)–(4.9) reduce to the nominal solution. Also, with uncertainty, the structure of the design equations remains very similar to the nominal case. In fact, if the transducer B/A is known, i.e. $\Delta\mathscr{G}=0$, we can substitute $\bar{C}\bar{C}_*$, $\tilde{M}\tilde{M}_*$, D, N for C_0C_{0*}, M_0M_{0*}, D_0, N_0 in (2.7)–(2.10), which then become identical to (4.3), (4.6)–(4.9). Robust and nominal filter design are then merged together into one pair of design equations, namely (2.8) and (2.9). (The reasoning in Grimble (1984) is based on similar substitutions.) This analogy cannot be generalized to uncertain transducers B/A, because of the presence of $\tilde{B}\tilde{B}_*$ in (4.3) and of $B_{0*}A_{1*}$ in (4.7). The criterion expressions (2.10) and (4.9) will also differ for $\Delta\mathscr{G}\neq 0$, due to the additional third term in (4.9).

Note that (4.9) consists of three terms. Term 1 represents the effect of finite smoothing lag m. As in Carlsson *et al.* (1991), it can be shown that $L_*(q)\to 0$ when $m\to\infty$. The second term mainly represents the effect of noise. It vanishes for $\rho=0$. Finally, the third term represents degradation caused by errors $\Delta\mathscr{G}$ in the transducer model. It vanishes only when $\Delta\mathscr{G}=0$. Note that if both ρ and $\Delta\mathscr{G}$ are zero, the impact of $\Delta\mathscr{F}$ diminishes as $m\to\infty$, because $LL_*\to 0$.

In situations with little noise and sufficiently large smoothing lag m, term 3 in (4.9) will dominate the error. This is not surprising; a deconvolution smoother then essentially inverts $\mathscr{G}$. This operation is sensitive to model errors there. Let $S=T=1$, $\rho=0$ and $m\to\infty$. Term 3 of (4.9) will then be $\bar{C}\bar{C}\bar{E}(\Delta\mathscr{G}\Delta\mathscr{G}_*)/\bar{E}(\mathscr{G}\mathscr{G}_*)DD_*$. It is proportional to the relative error in the spectrum of $\mathscr{G}$.

4.3. *Analytical expressions for performance evaluation*

Theorem 2. Let a nominal estimator $Q_0/R_0=Q_{10}N_0A_0/T\beta_0$ be designed by (2.7)–(2.9). Applying it, instead of (4.6), on an ensemble of systems results in an increase, compared to (4.9), of the mean MSE $\bar{E}E\,|z(t)|^2$. The increase is given by

$$\begin{aligned}\bar{E}E\,|z(t)|_0^2-\bar{E}E\,|z(t)|^2_{\min}&=\frac{r}{2\pi i}\oint\left|\frac{\beta}{DAN}\right|^2\left|\frac{Q_0}{R_0}-\frac{Q}{R}\right|^2\frac{dz}{z}\\&=\frac{r}{2\pi i}\oint\left|\frac{Q_{10}\beta-Q_1N_1A_1\beta_0}{D_0D_1A_1N_1T\beta_0}\right|^2\frac{dz}{z},\end{aligned}\tag{4.10}$$

where r, β is defined by (4.1)–(4.3) and Q/R is the optimal robust filter (4.6).

Proof. To obtain (4.10), the nominal filter Q_0/R_0 is expressed as $Q/R+(Q_0/R_0-Q/R)$. The optimality of Q/R implies that any modification of it gives an orthogonal contribution to the criterion. This, and the use of (4.1) gives the first equality of (4.10). Mixed terms vanish, due to the orthogonality. The second equality follows from (2.7) and (4.6).

Theorem 3. Let a robust estimator Q/R be designed by (4.3)–(4.7). When applying it on the nominal system, the increased MSE, compared to the minimum (2.10) is

$$\begin{aligned}E\,|z(t)|^2-E\,|z(t)|_0^2&=\frac{r_0}{2\pi i}\oint\left|\frac{\beta_0}{D_0A_0N_0}\right|^2\left|\frac{Q}{R}-\frac{Q_0}{R_0}\right|^2\frac{dz}{z}\\&=\frac{r_0}{2\pi i}\oint\left|\frac{Q_1\beta_0A_1N_1-Q_{10}\beta}{D_0T\beta}\right|^2\frac{dz}{z}.\end{aligned}\tag{4.11}$$

Proof. Analogous to Theorem 2, by expressing Q/R as $Q_0/R_0+(Q/R-Q_0/R_0)$.

Remarks. The middle expression in (4.10) can be used for arbitrary linear estimators Q_0/R_0, for example minimax-designs. Thus we do not have to evaluate the mean performance of alternative designs by Monte-Carlo simulation. The averaged innovations model β/DAN in (4.10), and the nominal innovations model $\beta_0/D_0A_0N_0$ in (4.11) can be seen as weighting functions. In frequency regions where their magnitude is large, differences between the two estimators will have a large impact on the performance.

4.4. *A numerical example.*

Example 3: *calculation and comparison of robust and nominal designs.* Consider an extended design model (3.3), with a FIR nominal model given by

$$\begin{aligned}C_0(q^{-1})=1-0.95q^{-1};\quad B_0(q^{-1})=0.5-0.4q^{-1};\\M_0(q^{-1})=1-0.8q^{-1},\quad k=1,\ \rho=0.001.\end{aligned}\tag{4.12}$$

The covariance matrices of ΔC, ΔB and ΔM are

$$\mathbf{P}_{\Delta C}=\begin{bmatrix}0&0\\0&0.040\end{bmatrix};\quad\mathbf{P}_{\Delta B}=\begin{bmatrix}0.0025&0\\0&0.0225\end{bmatrix};$$
$$\mathbf{P}_{\Delta M}=\begin{bmatrix}0&0\\0&0.010\end{bmatrix}.$$

The assumed standard deviations are thus 0.20 for c_1, 0.05 for b_0, 0.15 for b_1 and 0.10 for m_1.

We would like to obtain a robust filter $\hat{u}(t \mid t) = (Q/R)y(t)$. This estimator should, essentially, perform a one step prediction to obtain $u(t)$, since the transducer $q^{-k}B(q^{-1})$ has a one step delay. Using (4.5), we obtain

$$\begin{aligned} \bar{E}(\Delta C\, \Delta C_*) &= 0.040, \\ \bar{E}(\Delta B\, \Delta B_*) &= 0.0250, \\ \bar{E}(\Delta M\, \Delta M_*) &= 0.010. \end{aligned} \tag{4.13}$$

Inserting (4.13) into (4.3) gives

$$\begin{aligned} r\beta\beta_* = &(1.9425 - 0.95(q + q^{-1})) \\ &\times (0.435 - 0.2(q + q^{-1})) \\ &+ 0.001(1.65 - 0.8(q + q^{-1})). \end{aligned}$$

By solving for the stable monic $\beta(q^{-1})$ and r, we obtain

$$\begin{aligned} \beta(q^{-1}) &= 1 - 1.4659q^{-1} + 0.5315q^{-2}, \\ r &= 0.3575. \end{aligned}$$

Proceed to calculate the filter polynomial $Q_1(q^{-1})$ from (4.7), in which $S = A_1 = N_0 = N_1 = D_0 = D_1 = T = 1$. With degrees $nQ_1 = 0$, $nL = 2$, we obtain, with $Q_1(q^{-1}) = c$,

$$\begin{aligned} q(1.9425 &- 0.95(q + q^{-1}))(0.5 - 0.4q) \\ &= 0.3575(1 - 1.4659q + 0.5315q^2)c \\ &\qquad + q(\ell_0 + \ell_1 q + \ell_2 q^2). \end{aligned}$$

Equating for different powers of q gives

$$\begin{aligned} Q_1(q^{-1}) &= -1.3288, \\ L_*(q) &= 0.6549 - 0.9995q + 0.380q^2. \end{aligned}$$

The robust estimator (4.6) is $\hat{u}(t \mid t) = (Q_1/\beta)y(t)$, or

$$\hat{u}(t \mid t) = \frac{-1.3288}{1 - 1.4659q^{-1} + 0.5315q^{-2}}\, y(t). \tag{4.14}$$

It has poles in $z = 0.8086$ and $z = 0.6573$. The nominal filter polynomials are

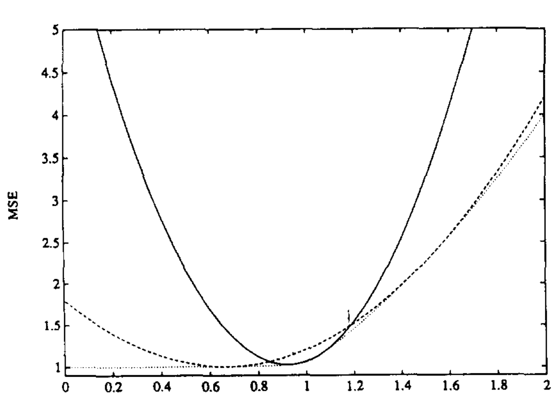
$$\begin{aligned} Q_{10}(q^{-1}) &= -1.8413, \\ \beta_0(q^{-1}) &= 1 - 1.7206q^{-1} + 0.7365q^{-2}, \end{aligned} \tag{4.15}$$

with estimator poles in $z = 0.9206$ and $z = 0.80$. The robust estimator has decreased the gain and moved the poles inward. It has become more cautious.

The performance of nominal and robust estimators is exemplified in Fig. 3. The MSE is much lower for the robust filter, for most parameter values. For a wide range of parameter variations, the performance of the robust estimator is close to that which could be obtained if the true parameters were known.

In Fig. 4, the mean performance (2.6) was calculated, for nominal and robust estimators, as a function of the standard deviations of some of the parameters. As expected, the mean performance is much better for the robust filter for large parameter deviations. In situations with small model errors, we might just as well use the nominal estimator. This is also true when the noise level is high; when $Ev(t)^2 \geq 0.1$, the difference between robust and nominal filters is very small.

In Fig. 5, we compare with minimax-designs, assuming two distributions, both with variances as above. (1) Rectangular (hard bound $\sqrt{3}\times$ standard deviations) and (2) five-point distributions of each parameter (bound ± 0.5 for c_1, b_1, m_1, ± 0.2 for b_0). Figure 5 clearly reveals the weaknesses of minimax designs: for wide model error distributions, with unlikely remote values, the filter performance for the nominal case deteriorates. More reasonable filters are obtained when the most remote value is close to the standard deviation of the distribution. However, even the assumption of a rectangular distribution results in a more conservative design



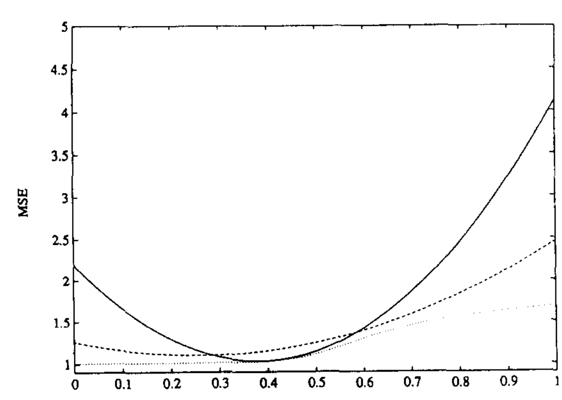


FIG. 3. Variation of one parameter in Example 3, while the others are nominal. The MSE $Ez(t)^2$, as a function of the zero $-c_1$ of the signal model (left) and of $-b_1$ in the transducer (right). Performance of the nominal filter (2.7) (solid), of the robust filter (4.6) (dashed) and the lower bound (dotted), achievable with the knowledge of the true parameter value.

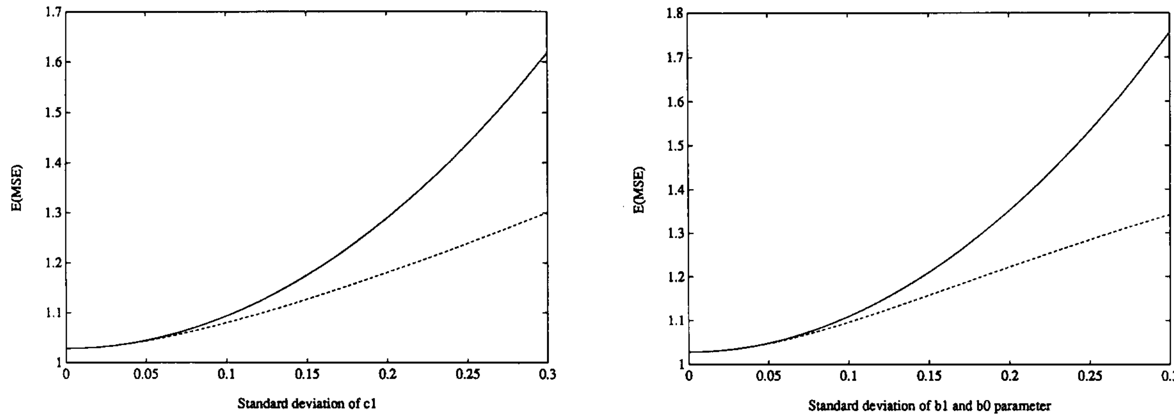


FIG. 4. Mean performance $\bar{E}Ez(t)^2$ in Example 3, when standard deviations of c_1 (left) and either b_0 or b_1 (right) are varied, while other coefficients are nominal. Nominal filters (solid) are compared to robust filters, designed for the corresponding parameter variance (dashed).

than our cautious Wiener filter. Additional details and comments are found in Appendix D.

The simplest way of robustifying an estimator is to just increase the noise variance ρ used in a nominal design. The performance of this technique is also illustrated in Fig. 5. The result around the nominal case is not as good as for our robust filter. The difference can be expected to be even larger in more high-order examples. The use of just one single robustification parameter mostly provides insufficient degrees of freedom. It can only vary β along a single root–locus trajectory.

4.5. *Averaging over systems vs averaging over models*

Conceptually, two types of "averaged MSE" could be defined.

Case 1. One filter is to be used on a class of different systems. The averaged performance $\bar{E}_{\mathrm{syst}}(\cdot)$, with respect to the systems, is then to be minimized.

Case 2. A set of filters, based on different design models, are all to be used on one (unknown) true system. The average performance $\bar{E}_{\mathrm{mod}}(\cdot)$, with respect to these models, is then of interest.

Our criterion (1.2) has been formulated with Case 1 in mind. Situations corresponding to Case 2 are, however, also encountered. An example is equalizer design for the pan-European digital mobile radio standard GSM. There, channel models of FIR type are estimated during repeated short training sequences, in which the input $u(t)$ is known. Estimated models are then used to reconstruct unknown symbols during other times. If neither transmitter nor receiver moves, the channel will be time-invariant during many training events, but the estimated channels will differ due to noise. Robustness of equalizer performance according to the criterion of Case 2 is then a reasonable design objective.

An attempt to design filters directly by minimizing $\bar{E}_{\mathrm{mod}}(E\,|z(t)|^2)$ does lead to intractable nonlinearities. We could, as a substitute,

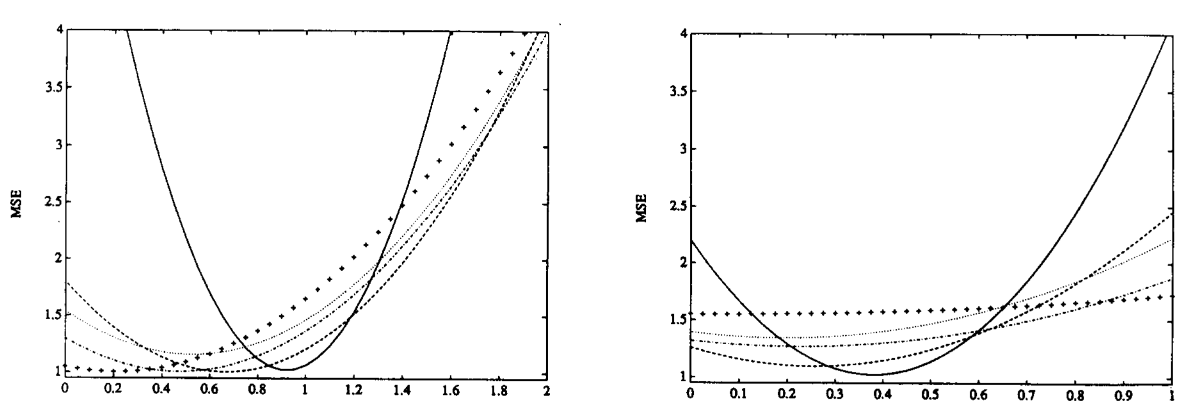


FIG. 5. Error variance $Ez(t)^2$ for nominal (solid) and robust (dashed) filters, as in Fig. 3. We also illustrate a minimax-design assuming rectangular distributions (dashed–dotted), a minimax-design based on five-point distributions (crosses) and a nominal filter, designed using $\rho = 0.1$ (dotted).

use our cautious Wiener filter (designed for Case 1) also in Case 2. For any nominal model, a filter would then be designed by minimizing $\bar{E}_{\text{syst}}(E\,|z(t)|^2)$ with respect to a (non-existing) set of systems around that particular model. Use of this procedure for all models would mostly, but not always, improve $\bar{E}_{\text{mod}}(E\,|z(t)|^2)$, compared to the use of nominal design. Exceptions exist, as illustrated by the following counter-example, suggested by Ari Kangas.

Example 3 (*continued*). The true system is assumed given by (4.12). One hundred different design models were generated, by adding random numbers $\Delta c_1 \in \mathcal{N}(0, 0.040)$, $\Delta b_0 \in \mathcal{N}(0, 0.0025)$, $\Delta b_1 \in \mathcal{N}(0, 0.0225)$ and $\Delta m_1 \in \mathcal{N}(0, 0.010)$ to the corresponding system coefficients. For each of the 100 models, robust and nominal filters were designed, as in Example 3. On average, nominal designs outperformed robust filtering! We obtained $\bar{E}_{\text{mod}}(E\,|z|^2) = 1.22$ for the robust filters and 1.11 for the nominal ones.

Fortunately, this kind of situation turns out to be a rather rare exception. The following Monte-Carlo simulation indicates that the cautious Wiener filter, on average, improves the performance similarly in Case 1 and Case 2.

Example 4: *performance for sets of randomly chosen models and systems*. We generated 500 different systems, with the structure

$$\begin{aligned} y(t) &= (b_0 + b_1 q^{-1} + b_2 q^{-2} + b_3 q^{-3} + b_4 q^{-4}) \\ &\quad \times u(t-1) + (1 - 0.8q^{-1})v(t), \\ u(t) &= \frac{1}{1 - 1.4q^{-1} + 0.65q^{-2}}\, e(t), \qquad (4.16) \\ Ee(t)^2 &= 1, \quad Ev(t)^2 = 0.001, \end{aligned}$$

by using independent random numbers $b_i \in \mathcal{N}(0, 1)$. The performance of filter estimators ($m = 0$) of $u(t)$ was investigated. In robust design, $\mathbf{P}_{\Delta B} = \text{diag}\,(0.04)$, $\mathbf{P}_{\Delta M} = 0$ and $\mathbf{P}_{\Delta C} = 0$ were used.

First, in an Experiment 1 illustrating Case 1, the 500 systems (4.16) represented different nominal design models. Around each of these 500 models, ensembles of true systems, with B-polynomials $B_0 + \Delta B$; $\sqrt{\bar{E}(\Delta b_i)^2} = 0.2$, were assumed to exist. Using (4.9) and (4.10), the mean performances, over each ensemble of true systems, were calculated for robust and nominal filters. We also calculated the minimal MSEs, achievable for known systems. The minimal MSEs varied between ≈ 1.0 and ≈ 6.0, depending on how well a particular transmission channel $B(q^{-1})$ could be inverted. Their average, over 500 cases, was 3.33. The average mean MSEs for robust and nominal design were

$$\frac{1}{500}\sum_{i=1}^{500} \bar{E}_{\text{syst}} E\,|z(t)|_i^2 \Big|_{\text{rob}} = 3.72,$$

$$\frac{1}{500}\sum_{i=1}^{500} \bar{E}_{\text{syst}} E\,|z(t)|_i^2 \Big|_{\text{nom}} = 6.25.$$

The improvement resulting from a robust design was rather small in ≈ 400 of the 500 cases, but could be very large in the other ones. This is illustrated by the left-hand histogram of Fig. 6. Models for which a nominal design is sensitive have zeros of $B_0(z^{-1})$ close to the unit circle in the low-frequency range, where $u(t)$ has significant energy (see Fig. 7).

Next, an Experiment 2 was performed to investigate Case 2. The 500 systems (4.16) here represented different true systems. For each of them, 100 design models were generated, by adding independent random coefficients $\Delta b_i \in$

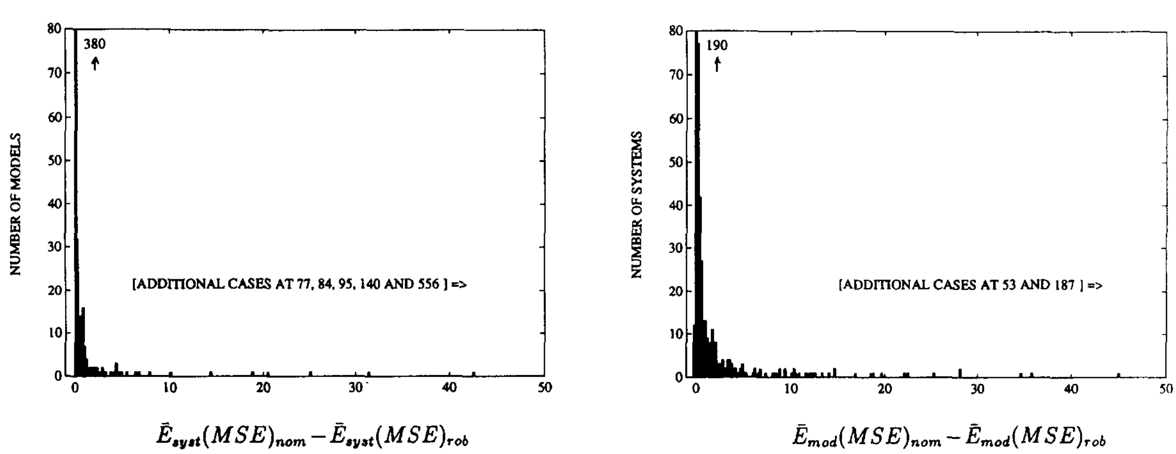


FIG. 6. Distribution of the reduction in mean MSE with robust, as compared to nominal, design in Example 4. In Experiment 1 (left), averages are taken over ensembles of systems, for each of 500 different nominal models of type (4.16). In Experiment 2 (right), we average over 100 models, for each of 500 true systems.

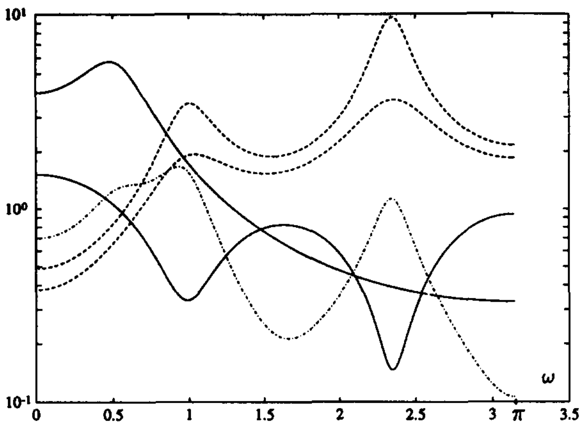


FIG. 7. Magnitude spectra in one particular case in Example 4, Experiment 1. Upper solid curve: input u. Lower solid curve: channel $B_0(e^{i\omega})$. Upper dashed: nominal filter. Lower dashed: robust filter. Dashed-dotted: frequency distribution of the mean RMS performance loss (square root of the integrand in (4.10)) with nominal filtering, if the system is uncertain.

$\mathcal{N}(0, 0.04)$ to the B-coefficients of (4.16). Based on these models, robust and nominal filters were designed. For each system, estimates of their average performance over the model distribution, $\bar{E}_{\text{mod}}(E\,|z(t)|^2)$, were calculated.† The total averages, over the 500 different systems, were

$$\frac{1}{500}\sum_{i=1}^{500} \bar{E}_{\text{mod}} E\,|z(t)|_i^2\bigg|_{\text{rob}} = 3.76,$$

$$\frac{1}{500}\sum_{i=1}^{500} \bar{E}_{\text{mod}} E\,|z(t)|_i^2\bigg|_{\text{nom}} = 6.01.$$

There were 18 cases out of 500 in which nominal designs, on average over 100 models, out-performed the robust ones. The performance difference was then small, in the range

$$0.01 \le \bar{E}_{\text{mod}}(\text{MSE})_{\text{rob}} - \bar{E}_{\text{mod}}(\text{MSE})_{\text{nom}} \le 0.22.$$

Typically, a robust design out-performed a nominal one. As in Experiment 1, the difference was mostly rather small, but was very large for a significant minority of systems. See Fig. 6. A detailed investigation of the experiments revealed a very reliable behavior of the robust filters, with mean MSEs seldomly more than 20% above MSEs obtainable with a known system. When a nominal design was sensitive, robust design eliminated the sensitivity, resulting in a greatly improved mean performance. No evidence of bad behaviour of robust filters was detected. No case of a large superiority of nominal design, as compared to robust design, were found in Experiment 2. To conclude, in situations described by either of Case 1 or Case 2, much could be gained, and little was lost, by using the cautious Wiener filter.

† For each of the 100 design models, a robust and a nominal filter was obtained. Calculation of the MSE for each of those filters, and averaging over the results based on 100 models, gave estimates of $\bar{E}_{\text{mod}} E\,|z(t)|^2\,|_{\text{rob}}$ and $\bar{E}_{\text{mod}} E\,|z(t)|^2\,|_{\text{nom}}$, respectively.

5. ROBUST STATE ESTIMATION

Using the principles discussed in the previous sections, a polynomial equations approach to robust state estimation can be developed. Consider a stable stochastic model in state-space form. It has n states, a scalar output $s(t)$ and an ℓ-vector $f(t)$, to be estimated:

$$x(t+1) = Fx(t) + e_v(t), \tag{5.1}$$

$$s(t) = H_s x(t); \quad f(t) = H_f x(t). \tag{5.2}$$

Here, $e_v(t)$ is white, with covariance matrix $R_v \ge 0$. If $\ell = n$ and $H_f = I_n$, full state estimation is desired. Measurements $y(t)$ of $s(t)$ are corrupted by coloured noise

$$y(t) = s(t) + \frac{M(q^{-1})}{N(q^{-1})} v(t), \tag{5.3}$$

with $E\,|v(t)|^2 \triangleq \rho$. Now, express (5.1) as

$$x(t) = \frac{1}{D(q^{-1})} \mathbf{C}(q^{-1}) e_v(t), \tag{5.4}$$

where $D(q^{-1}) = \det(I - q^{-1}F)$ is the characteristic polynomial and $\mathbf{C}(q^{-1}) = \operatorname{adj}(I - q^{-1}F)q^{-1}$ is a $n\,|\,n$ polynomial matrix. The $\ell\,|\,n$-matrix H_f is known. Other parts of the model (5.2)–(5.4) may be uncertain. We will consider an extended design model with the following structure.

$$H_s = H_0 + \Delta H, \tag{5.5}$$

$$\frac{M(q^{-1})}{N(q^{-1})} = \frac{M_0(q^{-1})}{N_0(q^{-1})} + \frac{M_1(q^{-1})\,\Delta M(q^{-1})}{N_1(q^{-1})}, \tag{5.6}$$

$$\frac{1}{D(q^{-1})}\mathbf{C}(q^{-1}) = \frac{1}{D_0(q^{-1})}\mathbf{C}_0(q^{-1}) + \frac{1}{D_1(q^{-1})}\mathbf{C}_1(q^{-1})\Delta\mathbf{C}(q^{-1}). \tag{5.7}$$

In (5.5), ΔH is a $1\,|\,n$-vector containing zero mean stochastic elements, with

$$\bar{E}(\Delta H^* \,\Delta H) \triangleq \mathbf{P}_H \ge 0.$$

The noise model (5.6), with covariance matrix $\mathbf{P}_{\Delta M} \ge 0$ and N_0, N_1 stable, is of the type introduced in Section 3. In (5.7), D_0 and D_1 are stable. The elements of $\Delta\mathbf{C}(q^{-1})$ are zero mean stochastic polynomials, defined as in (3.2). Different elements of $\Delta\mathbf{C}$ may be correlated, but $\Delta\mathbf{C}$ and ΔH are mutually uncorrelated. If elements of F in (5.1) are uncertain, series expansion gives a model of type (5.7). However, as was discussed in Example 1, a better description may be obtained for large uncertainties if $D_1(q^{-1})$, $\mathbf{C}_1(q^{-1})$ and $\Delta\mathbf{C}(q^{-1})$ are tuned to the uncertainty directly. Introduce the polynomial matrix

$$\tilde{\mathbf{P}} \triangleq \bar{E}(\mathbf{C}R_e\mathbf{C}_*) = D_1D_{1*}\mathbf{C}_0R_e\mathbf{C}_{0*} + D_0D_{0*}\mathbf{C}_1\bar{E}(\Delta\mathbf{C}R_e\,\Delta\mathbf{C}_*)\mathbf{C}_{1*}. \quad (5.8)$$

Elements of $\bar{E}(\Delta\mathbf{C}R_e\,\Delta\mathbf{C}_*)$ are linear combinations of polynomials of type (4.5).

Let H_{fi} denote the ith row of H_f. Introduce the following polynomials

$$\tilde{P}_{fi} \triangleq \bar{E}(H_{fi}\mathbf{C}R_e\mathbf{C}_*H_f^*) = H_{fi}\tilde{\mathbf{P}}H_0^*, \quad (5.9)$$

$$\tilde{P}_s \triangleq \bar{E}(H_s\mathbf{C}R_e\mathbf{C}_*H_s^*) = \mathrm{tr}\,[\tilde{\mathbf{P}}\bar{E}(H_s^*H_s)] = \mathrm{tr}\,[\tilde{\mathbf{P}}(H_0^*H_0 + \mathbf{P}_H)]. \quad (5.10)$$

Introduce the stable and monic average spectral factor $\beta(q^{-1})$, and a scalar r, from

$$r\beta\beta_* \triangleq \bar{E}\{H_s\mathbf{C}R_e\mathbf{C}_*H_s^*NN_* + \rho DD_*MM_*\} = \tilde{P}_sNN_* + \rho DD_*\tilde{M}\tilde{M}_*, \quad (5.11)$$

where $N = N_0N_1$, $D = D_0D_1$ and where $\tilde{M}\tilde{M}_*$ is defined as in (4.2). A stable β exists if and only if $\tilde{P}_s$ and $\rho\tilde{M}\tilde{M}_*$ have no common factors on $|z| = 1$.

Theorem 4. If a stable β from (5.11) exists, a stable causal estimator, which minimizes

$$\bar{E}E\,\mathrm{tr}\,[z(t)z(t)^*] = \bar{E}E\sum_{i=1}^{\ell}|f_i(t) - \hat{f}_i(t \mid t + m_i)|^2,$$

is

$$\hat{f}(t) = (q^{m_1}Q_1 \cdots q^{m_\ell}Q_\ell)^T\frac{N_0N_1}{\beta}y(t), \quad (5.12)$$

where the polynomials $Q_i(q^{-1})$, together with $L_{i*}(q)$, are unique solutions to

$$q^{-m_i}\tilde{P}_{fi}N_{0*}N_{1*} = r\beta_*Q_i + qD_0D_1L_{i*}, \quad i = 1 \cdots \ell, \quad (5.13)$$

with generic degrees

$$nQ_i = \max\{n\tilde{p}_{fi} + m_i,\ nd_0 + nd_1 - 1\},$$
$$nL_i = \max\{nn_0 + nn_1 - m_i,\ n\beta\} - 1.$$

Proof. See Appendix C.

Remarks. Note that the lags m_i may differ. After calculating (5.8)–(5.10), the design equations consist of the scalar spectral factorization (5.11) and of the ℓ uncoupled scalar Diophantine equations (5.13). (Since only the left-hand sides of (5.13) differ, they can be solved as a single system of linear equations.) For models without uncertainty, (5.12) is a transfer function formulation of a stationary Kalman estimator. For the nominal case with scalar $f(t)$, see Carlsson *et al.* (1991).

6. ROBUST FEEDFORWARD CONTROL

It is easy to utilize the proposed technique in order to improve the performance robustness of the LQG feedforward controllers discussed, for example, by Sternad and Söderström (1988) or Hunt (1989). Consider a stable system with scalar output to be controlled $y(t)$, input $u(t)$ and a measurable signal $w(t)$. With a similar notation as in (2.1)–(2.3), it is described by

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})}u(t-k) + \frac{D(q^{-1})}{F(q^{-1})}w(t-d),$$
$$w(t) = \frac{G(q^{-1})}{H(q^{-1})}v(t). \quad (6.1)$$

Here, $v(t)$ is white, with unit variance. As in Section 3, we use the structure

$$\frac{B}{A} = \frac{B_0}{A_0} + \frac{B_1\,\Delta B}{A_1},$$
$$\frac{D}{F} = \frac{D_0}{F_0} + \frac{D_1\,\Delta D}{F_1}, \quad (6.2)$$
$$\frac{G}{H} = \frac{G_0}{H_0} + \frac{G_1\,\Delta G}{H_1},$$

where ΔB, ΔD and ΔG are stochastic polynomials. They have zero means and known positive semidefinite autocovariance matrices $\mathbf{P}_{\Delta B}$, etc. All denominators are stable.

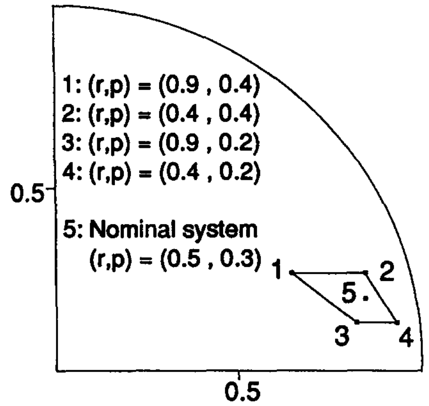
A stable and causal feedforward filter, operating on $w(t)$, is to be designed in order to minimize an infinite horizon quadratic criterion, with input penalty $\rho \geq 0$

$$J = \bar{E}E(|y(t)|^2 + \rho\,|u(t)|^2). \quad (6.3)$$

In a disturbance measurement feedforward problem, $w(t)$ represents the disturbance. In a command feedforward problem, $w(t)$ is a command signal and $(G/H)v(t)$ is a stochastic model, describing its second order properties. A servo filter is then to be designed, so that $(B/A)u(t-k)$ optimally follows a response model $-(D/F)w(t-d)$. The solution to both of these problems is presented below. First, calculate a stable and monic spectral factor $\tilde{G}(q^{-1})$ and a scalar s from

$$s\tilde{G}\tilde{G}_* = \bar{E}(GG_*) = G_0G_{0*}H_1H_{1*} + H_0H_{0*}G_1G_{1*}\bar{E}(\Delta G\,\Delta G_*). \quad (6.4)$$

It exists if and only if $\{G_0,\ G_1\bar{E}(\Delta G\,\Delta G_*)\}$ have no common zeros on $|z| = 1$. Then, calculate another stable and monic spectral factor $\beta(q^{-1})$,



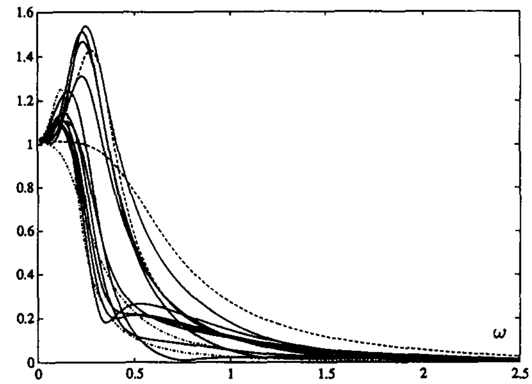


FIG. 8. Pole location (left) and Bode magnitude plot of the system B/A in Example 5. Four corner points $(r, p) = (0.9, 0.2)$, $(0.4, 0.2)$ (dashed-dotted), $(0.9, 0.4)$, $(0.4, 0.4)$ (dashed) are compared to 10 systems chosen randomly, using a rectangular distribution, from the set defined by the adjusted error model (solid).

and a scalar $\tilde{r}$, from

$$\tilde{r}\beta\beta_* = B_0B_{0*}A_1A_{1*} + \bar{E}(\Delta B\,\Delta B_*) \times B_1B_{1*}A_0A_{0*} + \rho A_0A_{0*}A_1A_{1*}. \quad (6.5)$$

It exists if either $\rho > 0$ or if $\{B_0,\ B_1\bar{E}(\Delta B\,\Delta B_*)\}$ lack common zeros on $|z| = 1$.

Theorem 5. A stable and causal feedforward filter minimizing (6.3) is given by

$$u(t) = -\frac{Q_1A_0A_1}{\beta F_0\tilde{G}}\,w(t), \quad (6.6)$$

where $Q_1(q^{-1})$, together with $L_*(q)$, is the unique solution to

$$q^{-d+k}B_{0*}A_{1*}D_0\tilde{G} = \tilde{r}\beta_*Q_1 + qF_0H_0H_1L_*, \quad (6.7)$$

with generic degrees

$$nQ_1 = \max(nd_0 + n\tilde{g} + d - k,\ nf_0 + nh_0 + nh_1 - 1),$$
$$nL = \max(nb_0 + na_1 - d + k,\ n\beta) - 1.$$

A direct proof closely follows the reasoning in Appendix B and is omitted here. The duality discussed in Bernhardsson and Sternad (1993) can also be used. The various transfer function uncertainties affect the design equations differently.

- The uncertainty ΔG of the disturbance or reference enters via (6.4). It affects the design in a similar way as would a measurement noise on $w(t)$.
- The uncertainty ΔB of the system enters via (6.5), in a similar way as does the input penalty ρ.† Also, note the presence of A_1 in (6.6). If $|B_1/A_1|$ is large at frequency ω_1, indicating large uncertainty, the filters has low gain at ω_1.
- Perhaps somewhat surprisingly, uncertainty in D/F has no effect at all on the regulator design. By duality (see Bernhardsson and Sternad, 1993), this corresponds to uncertainty in S/T having no effect in Section 4.

The intuitive notion that assumption of measurement noise on $w(t)$ and increase of the input penalty ρ would improve the performance robustness of a nominal design, is thus supported. With the above equations, the robustness can be tailored more exactly to the expected type and amount of uncertainty.

Example 5: *frequency domain adjustment of an error model and robustness improvement of a disturbance measurement feedforward regulator.* Assume the disturbance $w(t)$ in (6.1) to be white. The disturbance transfer function is $D/F = 1/(1 - 0.5q^{-1})$, with delay $d = 2$. No uncertainty in G/H or D/F is assumed, for simplicity. The transfer function

$$q^{-k}\frac{B}{A} = \frac{q^{-2}b_0(1 + 0.8q^{-1})}{(1 - z_1q^{-1})(1 - z_2q^{-1})},$$

has uncertain complex-conjugated poles

$$z_{1,2} = 1 - rp \pm i0.67p; \quad r \in [0.4, 0.9], \quad p \in [0.2, 0.4].$$

The static gain is assumed to be exactly $=1$, so we normalize to $b_0 = (r^2 + 0.67^2)p^2/1.8$. At higher frequencies, the uncertainty increases. Both the bandwith and the damping can vary significantly (see Fig. 8). Information about the exact distribution of r and p is not assumed to be available. (This is often the case in practice.) Still, a multiplicative error model

$$\frac{B}{A} = \frac{B_0}{A_0} + \frac{B_1\,\Delta B}{A_1}; \quad \frac{B_1}{A_1} = \frac{B_0}{A_0}\frac{B_m}{A_m},$$

† In fact, the effects of ρ and of $\bar{E}(\Delta B_*)$ are identical if $A_1 = 1$ and $B_1 = q^{-c}$, when $\mathbf{P}_{\Delta B}$ is diagonal. ($\bar{E}(\Delta B\,\Delta B_*)$ is then a scalar, cf. (4.5).)

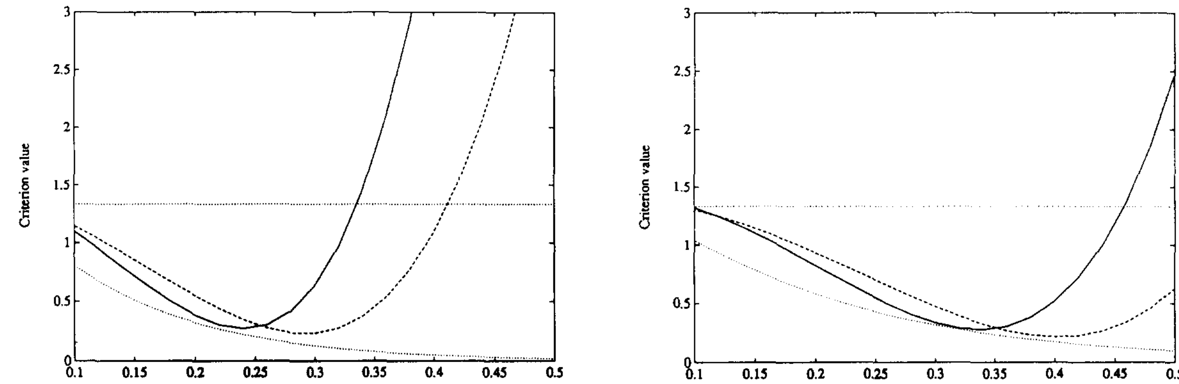


FIG. 9. Criterion value in Example 5 for well damped systems, $r = 0.9$ (left) and lightly damped systems, $r = 0.4$ (right). Use of the nominal controller (solid) is compared to use of the robust one (dashed), when the parameter p of the system is varied. Compare to the LQG criterion value achievable for a known system (lower dotted) and to the output variance without control (upper dotted).

can approximate the set of systems. Its properties are straightforward to adjust by inspection in the frequency domain, to roughly cover the expected range of variations in the dynamics. As nominal model, we select the model for $r = 0.5$, $p = 0.3$, which is in the central part of the area of pole locations:

$$\frac{B_0}{A_0} = \frac{0.035(1 + 0.8q^{-1})}{1 - 1.70q^{-1} + 0.7617q^{-2}}.$$

Since the relative uncertainty is low at low frequencies, we select a high-pass filter as fixed multiplicative error gain. A good fit is obtained with

$$\frac{B_m}{A_m} = 0.80\frac{(1 - 0.95q^{-1})^2}{(1 - 0.80q^{-1})^2}.$$

The covariance matrix $\mathbf{P}_{\Delta B}$ of the coefficients of ΔB is selected as $\mathrm{diag}\,(\alpha\lambda^i)$, as in Example 2, Section 3.2. Use of $\alpha = 0.5$ and $\lambda = 0.5$ gives a reasonable fit to the expected range of variations in the dynamics (see Fig. 8). Based on this model, nominal and robust feedforward filters were designed for the input penalty $\rho = 0.0001$. The robust filter was calculated from (6.4)–(6.7), with result $s = 1$, $\tilde{G} = 1$, $\tilde{r} = 0.00334$, $Q_1(q^{-1}) = 12.78$ and

$$\beta(q^{-1}) = A_0(q^{-1})(1 - 1.2490q^{-1} + 0.0146q^{-2} + 0.2634q^{-3} + 0.0146q^{-4}).$$

As is evident from Fig. 9, the sensitivity of a robust design is reduced significantly, as compared to the nominal regulator. In contrast, an attempt to perform a minimax design revealed that no saddle point solution exists in this example. Under such conditions, a minimax approach is extremely impractical.

7. CONCLUSIONS

Estimation and feedforward control, based on imperfectly known linear discrete-time models, has been considered. Model errors were represented as additive transfer functions with random numerators. A robust design was obtained by minimizing the squared estimation error, averaged both with respect to model errors and noise. This allows large but unlikely model errors to be taken into account, without dominating the design. The resulting filter becomes cautious, but not conservative.

With the presented polynomial equations approach, the robust filter design becomes simple and straightforward: just sum elements along diagonals of covariance matrices. Then, solve somewhat modified, "averaged", spectral factorizations and Diophantine equations. Application to related problems such as decision feedback equalization, discussed in Sternad and Ahlén (1990), is straightforward. See Sternad and Ahlén (1993). Nonsingular continuous-time problems can be solved, with obvious modifications of the equations. Multivariable design is under investigation.

In the open-loop problems considered here, filter stability could easily be assured. The use of stochastic error models in robust feedback design is more problematic. An interesting conceptual shift is that the notion of guaranteed robust stability mostly has to be abandoned, for unbounded (e.g. Gaussian) error distributions. Only stability with a certain probability can then be ascertained. See e.g. Stengel and Ray (1991). This may seem unappealing from a theoretical standpoint. However, it is the price to be paid for allowing infinite tail model error distributions.

There is a need for further work on systematic ways to estimate error models, both in the time domain and in the frequency domain. The method of Goodwin *et al.* (1991) requires known inputs. Parameter set identification for ARMA-models needs to be developed. A method for transforming an ordinary ARMA-model C_0/D_0, with covariance data on C_0 and D_0, into a well approximating extended design model of type $C_0/D_0 + C_1\,\Delta C/D_1$ is under development.

With very large system variations, the performance of even a robust linear filter will be unsatisfactory. An approach analogous to gain scheduling in feedback control can then be of use. A bank of filters is designed, with each filter attuned to a subset of the total system class. By using the output or auxiliary information, the most likely subset is selected, and the corresponding filter is used. See e.g. Lainiotis (1976) or Padilla and Haddad (1976). Robust design is a complement to this approach. By utilizing robust filters, which give acceptable behaviour for large model subsets, the number of filters in the filter bank may be reduced.

Robust design could also complement adaptation. Adaptive robust filtering/control, based on on-line estimation of nominal models and also of error model parameters, is a challenging subject. It is a main goal motivating our present research.

Acknowledgements—The authors wish to thank Ari Kangas, Vladimir Kučera, Kjell Nordström and Torbjörn Wigren for valuable comments on the manuscript.

REFERENCES

Ahlén, A. and M. Sternad (1989). Optimal deconvolution based on polynomial methods. *IEEE Trans. on Acoustics, Speech and Signal Processing*, **37**, 217–226.

Ahlén, A. and M. Sternad (1991). Wiener filter design using polynomial equations. *IEEE Trans. on Signal Processing*, **39**, 2387–2399.

Ahlén, A. and M. Sternad (1993). Optimal filtering problems. In K. Hunt (ed.), *Polynomial Methods in Optimal Control and Filtering*, Engineering Series, Peter Peregrinus, London.

Bernhardsson, B. and M. Sternad (1993). Feedforward control is dual to deconvolution. *Int. J. of Control* (to appear).

Carlsson, B., A. Ahlén and M. Sternad (1991). Optimal differentiation based on stochastic signal models. *IEEE Trans. on Signal Processing*, **39**, 341–353.

Carlsson, B., M. Sternad and A. Ahlén (1992). Digital differentiation of noisy data measured through a dynamic system. *IEEE Trans. on Signal Processing*, **40**, 218–221.

Chung, R.C. and P.R. Bélanger (1976). Minimum-sensitivity filter for linear time-invariant stochastic systems with uncertain parameters. *IEEE Trans. Aut. Control*, **21**, 98–100.

D'Appolito, J. A. and C. E. Hutchinson (1972). A minimax approach to the design of low sensitivity state estimators. *Automatica*, **8**, 599–608.

Fogel, E. and Y. F. Huang (1982). On the value of information in system identification—bounded noise case. *Automatica*, **18**, 229–238.

Goodwin, G. C., M. Gevers and B. Ninness (1992). Quantifying the error in estimated transfer functions with application to model order selection. *IEEE Trans. Aut. Control*, **37**, 913–928.

Goodwin, G. C. and M. E. Salgado (1989). A stochastic embedding approach for quantifying uncertainty in the estimation of restricted complexity models. *Int. J. of Adaptive Control and Signal Processing*, **3**, 333–356.

Goodwin, G. C., B. Ninness and H. V. Poor (1991). Choice of basis functions for continuous and discrete system modelling. *IFAC Symposium on Identification and System Parameter Estimation*, Budapest, Hungary, July 1991, 1179–1184.

Grimble, M. J. (1984). Wiener and Kalman filters for systems with random parameters. *IEEE Trans. Aut. Control*, **29**, 552–554.

Grimble, M. J. and A. ElSayed (1989). H_∞ robust linear estimator. *IFAC Symposium on Adaptive Control and Signal Processing*, **ACASP-89**, 159–164, Glasgow, U.K.

Haddad, W. F. and D. S. Bernstein (1988). Robust, reduced-order nonstrictly proper state estimation via the optimal projection equations with guaranteed cost bounds. *IEEE Trans. Aut. Control*, **33**, 591–595.

Hunt, K. J. (1989). *Stochastic Optimal Control Theory with Application to Self-tuning Control.* Springer-Verlag, Berlin.

Jazwinski, A. H. (1970). *Stochastic Processes and Filtering Theory.* Academic Press, New York.

Kassam, S. A. and T. L. Lim (1977). Robust Wiener filters. *J. of the Franklin Institute*, 171–185.

Kassam, S. A. and H. V. Poor (1985). Robust techniques for signal processing: a survey. *Proc. of the IEEE*, **73**, 433–481.

Kučera, V. (1979). *Discrete Linear Control. The Polynomial Equations Approach.* Wiley, New York.

Kučera, V. (1981). New results in state estimation and regulation. *Automatica*, **17**, 745–748.

Kučera, V. (1991). *Analysis and Design of Discrete Linear Control Systems.* Academia, Prague, Prentice Hall, London.

Lainiotis, D. G. (1976). Partitioning: a unifying framework for adaptive systems. Part i: estimation. *Proc. of the IEEE*, **64**, 1126–1143.

Leondes, C. T. and J. O. Pearson (1972). A minimax filter for systems with large plant uncertainties. *IEEE Trans. Aut. Control* **17**, 266–268.

Limebeer, D. J. N., B. D. O. Anderson and B. Hendel (1991). Mixed H_2/H_∞ filtering by the theory of Nash Games. Tokyo Workshop on Robust Control.

Ljung, L. (1987). *System Identification—Theory for the User.* Prentice Hall, Englewood Cliffs, NJ.

Loève, M. (1963). *Probability Theory.* D. Van Nostrand Company, New York.

Martin, C. J. and M. Mintz (1983). Robust filtering and prediction for linear systems with uncertain dynamics: a game-theoretic approach. *IEEE Trans. Aut. Control*, **28**, 888–896.

Moustakides, G. and S. A. Kassam (1983). Robust Wiener filters for random signals in correlated noise. *IEEE Trans. on Information Theory*, **29**, 614–619.

Moustakides, G. and S. A. Kassam (1985). Minimax equalization for random signals. *IEEE Trans. on Communications*, **33**, 820–825.

Nagpal, K. M. and P. P. Khargonekar (1991). Filtering and smoothing in an H_∞-setting. *IEEE Trans. Aut. Control*, **36**, 152–166.

Nahi, N. E. and E. J. Knobbe (1976). Optimal linear recursive estimation with uncertain system parameters. *IEEE Trans. Aut. Control*, **21**, 263–266.

Padilla, R. A. and A. H. Haddad (1976). On the estimation of uncertain signals using an estimation-detection scheme. *IEEE Trans. Aut. Control*, **21**, 509–512.

Poor, H. V. (1980). On robust Wiener filtering. *IEEE Trans. Aut. Control*, **25**, 531–536.

Speyer, J. L. and D. E. Gustafson (1975). An approximation method for estimation in linear systems with parameter uncertainty. *IEEE Trans. Aut. Control*, **20**, 354–359.

Stengel, R. F. and L. R. Ray (1991). Stochastic robustness of linear time-invariant control systems. *IEEE Trans. Aut. Control*, **36**, 82–87.

Sternad, M. and A. Ahlén (1990). The structure and design of realizable decision feedback equalizers for IIR channels with colored noise. *IEEE Trans. Information Theory*, **36**, 848–858.

Sternad, M. and A. Ahlén (1992). Robust decision feedback equalization. *ICASSP, Int. Conf. on Acoustics, Speech and Signal Processing*, Minneapolis, MN, 26–30 April, 1993.

Sternad, M. and T. Söderström (1988). LQG-optimal feedforward regulators. *Automatica*, **24**, 557–561.
Vastola, K. S. and H. V. Poor (1984). Robust Wiener–Kolmogorov theory. *IEEE Trans. on Information Theory*, **30**, 316–327.
Wahlberg, B. (1991). System identification using Laguerre models. *IEEE Trans. Aut. Control*, **36**, 551–562.
Wang, S. D., T. S. Kuo and C. F. Hsu (1987). Optimal observer design for linear dynamical systems with uncertain parameters. *Int. J. of Control*, **45**, 701–711.

APPENDIX A. PROOF OF LEMMA 2

Introduce a vector

$$f(\omega) \triangleq (1\ e^{i\omega} \cdots e^{i\omega dp})^T.$$

Then,

$$f^*(\omega)\mathbf{P}_{\Delta P} f(\omega) = g_{dp}^* e^{-i\omega dp} + \cdots + g_1^* e^{-i\omega} + g_0 + g_1 e^{i\omega} + \cdots + g_{dp} e^{i\omega dp}.$$

This real-valued scalar is the polynomial $\bar{E}(\Delta P\, \Delta P_*)$ from (4.5), evaluated on $z = e^{i\omega}$. Thus, since $\mathbf{P}_{\Delta C}$, $\mathbf{P}_{\Delta B}$ and $\mathbf{P}_{\Delta M}$ are all assumed positive semidefinite, the corresponding polynomials from (4.5) will be non-negative on the unit circle. This is a sufficient condition for the expressions (4.2) to be non-negative on the unit circle. If $M_1\bar{E}(\Delta M\, \Delta M_*)$ has no zeros on the unit circle, the same will then be true for $\tilde{M}\tilde{M}$: it can only have zeros on $|z| = 1$ which are common to $M_0 M_{0*}$ and $M_1 M_{1*}\bar{E}(\Delta M\, \Delta M_*)$. Such zeros are zeros also of $\tilde{C}\tilde{C}_*$ or $\tilde{B}\tilde{B}_*$ in (4.3) only if either of A3 or A4 are violated. This holds for the special case $\rho = 0$ as well.

APPENDIX B. PROOF OF THEOREM 1

In Ahlén and Sternad (1991), a novel technique for simple constructive derivation of polynomial design equations for Wiener filters is presented. It is based on the evaluation of orthogonality in the frequency domain. This technique will be utilized here, and be shown to be applicable to the criterion (2.6). If there is no model uncertainty, this derivation reduces to the derivation of (2.7)–(2.10).

With (2.1)–(2.3), the estimation error is

$$z(t) = \left(\frac{S}{T} - q^{m-k}\frac{Q}{R}\frac{B}{A}\right)\frac{C}{D}e(t) - q^m \frac{QM}{RN}v(t). \qquad \text{(B.1)}$$

All admissible alternatives to a proposed estimate $\hat{f}(t \mid t+m)$, given by (2.4), can be described by

$$\hat{d}(t) = (Q/R)y(t+m) + n(t),$$
$$n(t) = \mathcal{M}y(t+m).$$

Here, $\mathcal{M}(q^{-1})$ is a causal, stable but otherwise arbitrary rational transfer function. Optimality of (2.4) is obtained if no perturbation $n(t)$ can improve the mean estimator performance. This occurs if and only if the corresponding error $z(t)$ is orthogonal to any admissible estimator variation $n(t)$, i.e. $\bar{E}Ez(t)n^*(t) = \bar{E}Ez(t)^*n(t) = 0$. Then, the perturbed criterion value reduces to

$$\begin{aligned}\bar{E}E\,|f(t) - \hat{d}(t)|^2 &= \bar{E}E(|z(t)|^2 - z(t)n^*(t) \\ &\quad - z^*(t)n(t) + |n(t)|^2) \\ &= \bar{E}E(|z(t)|^2 + |n(t)|^2), \qquad \text{(B.2)}\end{aligned}$$

which is obviously minimized by $n(t) = 0$. Since all systems included in the extended design model (2.1)–(2.3) are assumed stable, both $z(t)$ and $n(t)$ will be stationary. Parseval's formula can then be used, to express $\bar{E}Ez(t)n^*(t)$ as

$$\begin{aligned}&\bar{E}E\left\{\left(\frac{S}{T} - q^{m-k}\frac{QB}{RA}\right)\frac{C}{D}e(t) - q^m\frac{QM}{RN}v(t)\right\} \\ &\times\left\{q^{m-k}\mathcal{M}\frac{BC}{AD}e(t) + q^m\mathcal{M}\frac{M}{N}v(t)\right\}^* = \bar{E}\frac{1}{2\pi i}\oint_{|z|=1} \\ &\times\left\{\left(z^{k-m}\frac{S}{T} - \frac{QB}{RA}\right)\frac{C}{D}\frac{B_*C_*}{A_*D_*} - \rho\frac{QMM_*}{RNN_*}\right\}\mathcal{M}_*\frac{dz}{z} \qquad \text{(B.3)}\end{aligned}$$

We are allowed to move the expectation $\bar{E}$ inside the integration, since, for any particular realization of ΔC, ΔB and ΔM, the integrand is Riemann integrable on the unit circle. See e.g. Jazwinski (1970). The expectation $\bar{E}$ operates on the numerators, since stochastic variables are present there only. Using the spectral factorization (4.1), and the fact that none of S, N, A, R, T or Q contain stochastic variables, (B.3) can be expressed as

$$\frac{1}{2\pi i}\oint\frac{(z^{k-m}SNN_*AR\bar{E}(CC_*B_*) - TQr\beta\beta_*)}{DTNARN_*A_*D_*}\mathcal{M}_*\frac{dz}{z}. \qquad \text{(B.4)}$$

Now, $\bar{E}Ez(t)n^*(t) = 0$ is fulfilled if all poles in $|z| < 1$ of the integrand are cancelled by zeros. Since $\mathcal{M}$ and NAD are stable, $(1/N_*A_*D_*)\mathcal{M}_*$ will have poles only in $|z| > 1$. All other poles are in $|z| < 1$. Thus, we require

$$z^{k-m}SNN_*AR\bar{E}(CC_*B_*) - TQr\beta\beta_* = zL_*DTNAR,$$

for some polynomial $L_*(z)$ or, equivalently,

$$(z^{k-m}SN_*\bar{E}(CC_*B_*) - zL_*DT)NAR = Qr\beta_*\beta T. \qquad \text{(B.5)}$$

The right-hand side of (B.5) must contain R as a factor. Since R must be stable, its factors cannot include factors of β_*. Thus, $\beta T = RH$ for some stable $H(z^{-1})$. Now, cancel R in (B.5). Observe that NA must be factor of QH, i.e. $QH = Q_1NA$. The filter $Q/R = (Q_1NA/H)/(\beta T/H)$ is (4.6). Cancel NA and substitute q for z in (B.5) to obtain

$$q^{k-m}SN_*\bar{E}(CC_*B_*) = r\beta_*Q_1 + qDTL_*. \qquad \text{(B.6)}$$

This is (4.7), since by using independence

$$\begin{aligned}\bar{E}(CC_*B_*) &= \bar{E}(CC_*)\bar{E}(B_0A_1 + A_0B_1\,\Delta B)_* \\ &= \tilde{C}\tilde{C}_*B_{0*}A_{1*}. \qquad \text{(B.7)}\end{aligned}$$

The "only if" part of the result follows because choices of Q/R other than (4.6) correspond to $n(t) \neq 0$, which, according to (B.2), increase the criterion value.

Remark on the degrees (4.8). Diophantine equations in general have an infinite number of solutions. In (4.7), however, causality requires Q_1 to be a polynomial only in q^{-1}, while optimality requires L_* to be a polynomial in q.† Their generic degrees (4.8) are then uniquely determined by the requirement that the highest occuring powers of q^{-1} and q, respectively, must be covered by the variables in (4.7). This gives an equal number of equations and unknowns. The linear system of equations is nonsingular, since β_* and DT have no common factors.‡

The expression (4.9). The minimal mean MSE is derived by inserting (B.1), (4.2), (4.6) and (B.6), in this order, into (2.6). This gives, after some calculations, the expression

$$\begin{aligned}\bar{E}E\,|z(t)|^2_{\min} &= \frac{1}{2\pi i}\oint\frac{SS_*\bar{E}(CC_*)r\beta\beta_* + DD_*TT_*LL_*}{r\beta\beta_*TT_*DD_*} \\ &\quad - \frac{SS_*NN_*\bar{E}(CC_*B_*)\bar{E}(CC_*B)}{r\beta\beta_*TT_*DD_*}\frac{dz}{z} \\ &= \frac{1}{2\pi i}\oint\frac{LL_*}{r\beta\beta_*} \\ &\quad + \frac{SS_*\tilde{C}\tilde{C}_*\{r\beta\beta_* - \tilde{C}\tilde{C}_*\tilde{B}\tilde{B}_*NN_* + \tilde{C}\tilde{C}_*(\tilde{B}\tilde{B}_* - B_0B_{0*}A_1A_{1*})NN_*\}}{r\beta\beta_*DD_*TT_*}\frac{dz}{z} \qquad \text{(B.8)}\end{aligned}$$

In the last equality of (B.8), we used $\bar{E}(CC_*B_*) = \tilde{C}\tilde{C}_*B_{0*}A_{1*}$ and exchanged

$$B_0B_{0*}A_1A_{1*} \quad \text{for} \quad B_0B_{0*}A_1A_{1*} + \tilde{B}\tilde{B}_* - \bar{B}\bar{B}_*.$$

† If L_* were allowed to have negative powers of z as arguments, poles at the origin would be introduced in the integrand of (B.4). The path integral would then not vanish.

‡ A more general discussion of these points can be found in Section IV of Ahlén and Sternad (1991, 1993), in Ahlén and Sternad (1993) and in Kučera (1979, 1991).

Now, utilizing (4.3) and, from (4.2),

$$\tilde{B}\tilde{B}_* - B_0B_{0*}A_1A_{1*} = A_0A_{0*}B_1B_{1*}\bar{E}(\Delta B\,\Delta B_*),$$

we obtain (4.9) by using $B_1\,\Delta B = \Delta\mathscr{G}A_1$.

The case without uncertainty. With $\Delta C = \Delta B = \Delta M = 0$, the spectral factorization (4.3) gives $r = r_0$ and $\beta = \beta_0 D_1 A_1 N_1$. Equation (4.7) then becomes

$$q^{-m+k}SC_0C_{0*}D_1D_{1*}B_{0*}A_{1*}N_{0*}N_{1*} = r_0\beta_{0*}D_{1*}A_{1*}N_{1*}Q_1 + qD_0D_1TL_*.$$

It is evident that D_1 must be a factor of Q_1 and $D_{1*}A_{1*}N_{1*}$ a factor of L_*. With $Q_1 = Q_{10}D_1$ and $L_* = L_{0*}D_{1*}A_{1*}N_{1*}$ and common factors cancelled, the above equation reduces to (2.9). The filter (4.6) equals (2.7).

APPENDIX C. PROOF OF THEOREM 4

The derivation follows the reasoning in the proof of Theorem 1 closely. Consider estimation of a single component $f_i(t)$ of $f(t)$, with a scalar $\hat{f}_i(t) = (Q/R)y(t+m_i)$. Use the orthogonality relation

$$\bar{E}Ez(t)n(t)^* = \bar{E}E(f_i(t) - (Q/R)y(t-m_i))(\mathscr{M}y(t+m_i))^* = 0.$$

Use of (5.2)–(5.4) and of (5.8)–(5.11) gives

$$\bar{E}Ez(t)n(t)^* = \bar{E}E\left\{\frac{[RH_{fi}\mathbf{C} - q^{m_i}QH_s\mathbf{C}]}{RD}e_v(t) - q^{m_i}\frac{Q}{R}\frac{M}{N}v(t)\right\}$$
$$\times\left\{\mathscr{M}q^{m_i}\left(\frac{H_s\mathbf{C}}{D}e_v(t) + \frac{M}{N}v(t)\right)\right\}^* = \bar{E}\frac{1}{2\pi i}\oint$$
$$\times\left(\frac{RH_{fi}\mathbf{C}R_e\mathbf{C}_*H_s^*z^{-m_i}}{RDD_*} - \frac{QH_s\mathbf{C}R_e\mathbf{C}_*\mathbf{H}_s^*}{RDD_*} - \frac{Q}{R}\frac{MM_*}{NN_*}\rho\right)$$
$$\times\mathscr{M}_*\frac{dz}{z} = \frac{1}{2\pi i}\oint\frac{(R\tilde{P}_{fi}z^{-m_i}NN_* - Qr\beta\beta_*)}{RDD_*NN_*}\mathscr{M}_*\frac{dz}{z}. \qquad (C.1)$$

All poles inside $|z| = 1$ of the integrand of (C.1) are cancelled by zeros if and only if

$$R\tilde{P}_{fi}z^{-m_i}NN_* - Qr\beta\beta_* = zRDNL_*,$$

for some polynomial $L_*(q)$ or, with $z \leftrightarrow q$,

$$R(\tilde{P}_{fi}q^{-m_i}N_* - qL_*D)N = Qr\beta\beta_*. \qquad (C.2)$$

We see that β must have R as a factor, i.e. $RH = \beta$ for some stable $H(q^{-1})$. Furthermore, N must be a factor of QH, i.e. $QH = Q_iN$. The scalar filter $q^{m_i}Q/R = (q^{m_i}Q_iN/H)/(\beta/H)$ can now be seen as row i of (5.12). Cancelling of NR in (C.2) reduces it to the ith of the equations (5.13). Estimation of component i of $f(t)$ does not affect the estimate of other components. The total estimator of $f(t)$ can be obtained as ℓ parallel scalar estimators of $f_i(t)$, derived as above. Thus, we obtain (5.12).

APPENDIX D. ADDITIONAL COMMENTS ON EXAMPLE 3

In Fig. 3, it is interesting to note that the robust filter attains its minimal estimation error for parameter values other than the nominal. The asymmetry of the ideal performance surface in parameter space (the dotted curves in Fig. 3) seems to be a main cause of the displacement.

In the minimax design, filters were constructed for the worst case (giving highest MSE). When these filters were applied to other systems in the class, the MSE was never higher than in the design case. Thus, the minimax solution had been found. The worst case for the rectangular distribution was

$$C(q^{-1}) = 1 - 1.3q^{-1}, \quad B(q^{-1}) = 0.41 - 0.66q^{-1},$$
$$M(q^{-1}) = 1 - 0.63q^{-1}.$$

The worst case for the five-point distribution was

$$C(q^{-1}) = 1 - 1.45q^{-1}, \quad B(q^{-1}) = 0.3 - 0.9q^{-1},$$
$$M(q^{-1}) = 1 - 0.3q^{-1}.$$

We feel that the comparison with minimax design is reasonably fair; both distributions, in particular the rectangular one, have considerable weights far from their mean values. One could easily have chosen other, more extreme, distributions, for which parameter values close to the worst case are much more unlikely.