# The Structure and Design of Realizable Decision Feedback Equalizers for IIR Channels with Colored Noise

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*Abstract* —**A simple algorithm for optimizing decision feedback equalizers (DFE) by minimizing the mean-square error (MSE) is presented. A complex baseband channel and correct past decisions are assumed. The dispersive channel may have infinite impulse response and the noise may be colored. We consider optimal realizable (stable and finite-lag smoothing) forward and feedback filters in discrete time. They are parameterized as recursive filters. In the special case of transmission channels with finite impulse response and autoregressive noise, the minimum MSE can be attained with transversal feedback and forward filters. In general, the forward part should include a noise-whitening filter (the inverse noise model). The finite realizations of the filters are calculated using a polynomial equation approach to the linear quadratic optimization problem. The equalizer is optimized essentially by solving a system of linear equations $Ax = B$, where $A$ contains transfer function coefficients from the channel and noise model. No calculation of correlations are required with this method. A simple expression for the minimal MSE is presented. The DFE is compared to MSE-optimal linear recursive equalizers. Expressions for the equalizer in the limiting case of infinite smoothing lags are also discussed.**

## I. Introduction

When digital data is sent over a noisy communication channel, intersymbol interference limits the achievable transmission rate. The intersymbol interference becomes severe when the symbol rate exceeds the nominal bandwidth of the channel. This problem also occurs for example in digital radio communication with multipath propagation. Equalizers are placed at the receiver to reconstruct the transmitted sequence [21]. Linear equalizers are one alternative. They consist of a linear filter in front of a nonlinear decision element [15], [17], [18]. The performance attained by linear equalization is often unsatisfactory. Signal energy may be placed within strongly attenuated parts of the transmission spectrum. The linear filter, being an approximate inverse of the channel, can then only reconstruct the transmitted signal at the expense of a large noise amplification.

Much higher performance can be attained by nonlinear equalizers. The best result is achieved by maximum likelihood estimation (MLSE) of entire data sequences. The MLSE Viterbi algorithm [16] however becomes prohibitively complex for channels with long impulse responses. There has been considerable work on modified MLSE schemes for channels with long or infinite impulse responses, see [25]–[28]. The goal has been to reduce the computational complexity, without too much performance degradation.

The decision feedback equalizer (DFE) is a very simple symbol-by-symbol detector. For many channels, it attains almost the same performance as the Viterbi equalizer [20]. Robustness against phase jitter has been found to be better for DFE, compared to MLSE [19].

A DFE consists of two linear filters and a decision nonlinearity (see Fig. 1). Previous symbol estimates are fed through a linear "feedback filter." Its output is subtracted from the present estimate, before it enters the decision module. The subtraction of all intersymbol interference caused by past symbols can be achieved, if past decisions were correct. The result is an equalizer that attains channel inversion with much less noise enhancement than a linear equalizer.

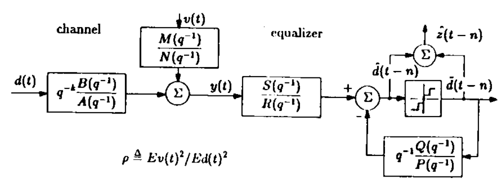


Fig. 1. Equalization problem. Transmitted sequence $d(t)$, propagating through channel and disturbed by noise, yields received sequence $y(t)$. Decision feedback equalization is used in order to restore $d(t)$. Stable IIR-feedforward and feedback filters are considered. $R = 1$ and $P = 1$ correspond to conventional DFE, and $Q = 0$ to linear recursive equalizer.

We will discuss the design of realizable filters in discrete time for a DFE. The mean square error (MSE) criterion is minimized and correct past decisions are assumed. The equalizer uses fixed-lag smoothing, i.e., estimation of symbol $d(t-n)$ based on measurements up to time $t$. The optimization of DFEs, using the MSE criterion, has been discussed repeatedly for the past 20 years [2]–[5]. (The zero forcing criterion [2] has been another design tool.) These works have treated optimization without the constraint of realizability. The results are non-

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1) Solve for the coefficients of the polynomials $S_1(q^{-1})$ and $\bar{L}_1(q^{-1})$ in

$$\left[\begin{array}{ccc|ccc} \tau_0 & & 0 & 1 & & 0 \\ \vdots & \ddots & & \gamma_1 & \ddots & \\ \tau_{n-k} & \cdots & \tau_0 & \gamma_{n-k} & \cdots \gamma_1 & 1 \\ \hline \rho & \rho\gamma_1^* \cdots & \rho\gamma_{n-k}^* & -\tau_0^* & \cdots & -\tau_{n-k}^* \\ & \ddots & \rho\gamma_1^* & & \ddots & \vdots \\ 0 & & \rho & 0 & & -\tau_0^* \end{array}\right] \cdot \left[\begin{array}{c} s_0 \\ \vdots \\ s_{n-k} \\ \hline l_{n-k}^* \\ \vdots \\ l_0^* \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]. \tag{3.6}$$

2) With $\{s_i\}$ and $\{l_j^*\}$ obtained from Step 1, perform the multiplication

$$\left[\begin{array}{ccc|ccc} \tau_0 & & 0 & 1 & & 0 \\ \vdots & \ddots & & \gamma_1 & \ddots & \\ \vdots & & \tau_0 & \vdots & & 1 \\ \tau_{\delta\tau} & & \vdots & \gamma_{\delta\gamma} & & \gamma_1 \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & & \tau_{\delta\tau} & 0 & & \gamma_{\delta\gamma} \end{array}\right] \cdot \begin{bmatrix} s_0 \\ \vdots \\ s_{n-k} \\ l_{n-k}^* \\ \vdots \\ l_0^* \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \alpha_1 \\ \vdots \\ \alpha_{\delta\alpha} \end{bmatrix} \tag{3.7}$$

yielding the coefficients of the polynomial $\alpha(q^{-1})$.

3) Calculate the polynomial $Q(q^{-1})$ from (3.1)

$$Q(q^{-1}) = q\left(\alpha(q^{-1}) - \gamma(q^{-1})\right). \tag{3.8}$$

The equivalent equalized channel (from $d(t)$ to $\hat{d}(t-n|t)$) will then be

$$\begin{aligned} C_{eq} &= q^{-k}\frac{BNS_1}{AM} - q^{-n-1}\frac{Q}{AM} \\ &= q^{-n} - q^{-k}\bar{L}_1(q^{-1}). \end{aligned} \tag{3.9}$$

*Proof:* See Appendix B. □

*Remarks:* Note that the matrix blocks in (3.6) are quadratic. If $\tau$ or $\gamma$ are of order $< n-k$, zeros are used to fill up the rows of the blocks in (3.6). The second step just represents calculation of $\alpha$ from equation (3.3a), with known $S_1$ and $\bar{L}_1$: $\tau S_1 + \gamma \bar{L}_1 = q^{-n+k}\alpha$. The polynomial $L_{2*}$, which is not needed in the filter, will be defined uniquely. It could be calculated from (3.3b).

With small modifications, the same calculations can be used for nonwhite data sequences. Let $d(t)$ be modeled by $d(t) = C(q^{-1})e(t)$, where $e(t)$ is white and $C(q^{-1})$ is monic and stable. Such a correlation could be introduced by channel coding or by the use of controlled intersymbol interference. The factor $\rho$ is redefined as $E|v(t)|^2/E|e(t)|^2$. It can then be shown that $P = MAC$, $\alpha = MAC + q^{-1}Q$ and $\tau = NBC$, inserted in the previous equations, give an optimal equalizer. The restriction that $C$ is stable is important. Note that $C$ would be a factor of the feedback denominator $P$.

An important question is if a unique solution to (3.6) can always be found without any restrictions on, for example, the coprimeness of $\tau(q^{-1})$ and $\gamma(q^{-1})$.

*Theorem 3:* Let the leading coefficient of $B$ be $b_0 \neq 0$. Then, (3.6) will always have a unique solution, $(S_1, \bar{L}_1)$.

*Proof:* See Appendix C. □

*Remarks:* When both $|b_0| = |\tau_0|$ and $\rho$ are small, the system (3.6) may be badly conditioned. If $\tau = BN$ and $\gamma = AM$ contain common factors, the optimal feedback filter, calculated from (3.2) and Theorem 2, will also contain these (stable) common factors.[1] Such factors can be cancelled before implementation. Hence, the remark about coprime factors in the formulation of Theorem 1.

Summing up, one can conclude that an equalizer can be calculated using (3.2) and (3.6)–(3.8) (Theorem 2). This procedure always works (Theorem 3). The resulting equalizer is MSE-optimal (Theorem 1). The minimal criterion value, assuming correct past decisions, is given by (3.5).

When considering its possible use as part of an adaptive equalizer, a drawback of the algorithm presented in Theorem 2 is that the order of the linear system, $2(n-k+1)$, is unnecessarily large. (There would, however, be no need to recalculate the equalizer for every sample.) By combining equation (3.6) with (3.9), it becomes evident that we actually need to solve a linear system of only half this size. The required number of multiplications is then reduced by a factor of 3–4 for typical values of $n$.

Let $\{h_i\}_0^\infty$ be the impulse response of $\tau(q^{-1})/\gamma(q^{-1})$:

$$\frac{\tau(q^{-1})}{\gamma(q^{-1})} = \frac{B(q^{-1})N(q^{-1})}{A(q^{-1})M(q^{-1})} = \sum_{i=0}^{\infty} h_i q^{-i}. \tag{3.10}$$

Equation (3.9) provides a relation between $S_1$, $L_1$ and the impulse response of the equalized channel

$$\begin{pmatrix} C_{eq_k} \\ \vdots \\ C_{eq_n} \end{pmatrix} = \begin{pmatrix} h_0 & & 0 \\ \vdots & \ddots & \\ h_{n-k} & \cdots & h_0 \end{pmatrix} \begin{pmatrix} s_0 \\ \vdots \\ s_{n-k} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} l_{n-k}^* \\ \vdots \\ l_0^* \end{pmatrix}. \tag{3.11}$$

Substitution of $\{l_i^*\}$ from (3.11) into the lower blocks of (3.6) results in the following linear system of order $n-k+1$

$$\left( \begin{bmatrix} \rho & \cdots & \rho\gamma_{n-k}^* \\ & \ddots & \vdots \\ 0 & & \rho \end{bmatrix} + \begin{bmatrix} \tau_0^* & \cdots & \tau_{n-k}^* \\ & \ddots & \vdots \\ 0 & & \tau_0^* \end{bmatrix} \cdot \begin{bmatrix} h_0 & & 0 \\ \vdots & \ddots & \\ h_{n-k} & \cdots & h_0 \end{bmatrix} \right) \begin{pmatrix} s_0 \\ \vdots \\ s_{n-k} \end{pmatrix} = \begin{pmatrix} \tau_{n-k}^* \\ \vdots \\ \tau_0^* \end{pmatrix}. \tag{3.12}$$

[1]Factors common to $\tau$ and $\gamma$ must also be factors of $\alpha$, according to (3.3a). From (3.8) it is evident that they then also are factors of $Q$. Thus, they will be common factors in the transfer function $Q/P = Q/\gamma$.

white, zero-mean sequence with equally probable values $\{-m+1,\cdots,-1,+1,\cdots,m-1\}$. In other modulation schemes, such as QAM, the model coefficients and signals in (2.1) are complex-valued. Define

$$\lambda_d \triangleq E|d(t)|^2, \qquad \rho \triangleq E|v(t)|^2/E|d(t)|^2. \quad (2.2)$$

The data sequence $d(t)$ is to be reconstructed from measurements of $y(t)$. As an estimator, introduce the following general IIR decision feedback equalizer (GDFE):

$$\hat{d}(t-n|t) = \underbrace{\frac{S(q^{-1})}{R(q^{-1})}}_{\text{forward filter}} y(t) - q^{-1} \underbrace{\frac{Q(q^{-1})}{P(q^{-1})}}_{\text{feedback filter}} \tilde{d}(t-n). \quad (2.3)$$

$n$ is the number of lags (smoothing lag) and $\tilde{d}(t-n)$ is decisioned data, for example, sign($\hat{d}(t-n)$) when PAM is used with $m=2$ (see Fig. 1). The denominator polynomials $R(q^{-1})$ and $P(q^{-1})$ are assumed to be monic, and required to be stable. The sampling rate is assumed to equal the symbol rate. (Fractionally-spaced equalizers are not considered in this paper.) If $d(t)$ is complex-valued, the coefficients of the filters must be complex.

Given a received sequence $y(t)$, a model (2.1), (2.2) and a smoothing lag $n \geq k$, the problem treated is to find polynomials $(S,R,Q,P)$ that minimize the mean-square estimation error (mse):

$$E|z(t-n)|^2 \triangleq E|d(t-n) - \hat{d}(t-n|t)|^2. \quad (2.4)$$

Because of the presence of a nonlinear decision circuit, it is impossible to get closed-form expressions for an optimal estimator. As in most previous treatments of the DFE problem, we will simplify the problem by *assuming correct past decisions*. This transforms the problem into a linear quadratic optimization problem, as shown in Fig. 2.

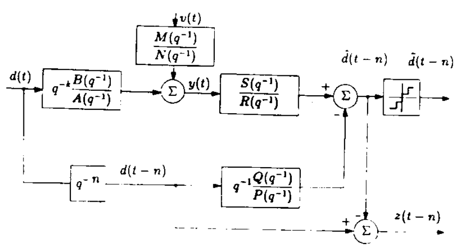


Fig. 2. Equalization problem. Provided past decisions are correct, structure in Fig. 1 can be transformed to this equivalent structure, where the decision nonlinearity is no longer present.

## III. The Optimal IIR-Decision Feedback Equalizer

We make the following definitions. Let $p_j^*$ denote a complex conjugate of the polynomial coefficient $p_j$. Define, for any polynomial $P(q^{-1}) = p_0 + p_1 q^{-1} + \cdots + p_{\delta p} q^{-\delta p}$,

$$P_* \triangleq p_0^* + p_1^* q + \cdots + p_{\delta p}^* q^{\delta p}$$

$$\bar{P} \triangleq q^{-\delta p} P_* = p_{\delta p}^* + p_{\delta p-1}^* q^{-1} + \cdots + p_0^* q^{-\delta p}.$$

When appropriate, the complex variable $z$ is substituted for the forward shift operator $q$. Polynomials $P(q^{-1})$ are called *stable* if all zeros of $P(z^{-1})$ are located in $|z|<1$. For convenience, the polynomial arguments will often be omitted. We introduce the following polynomials:

$$\tau(q^{-1}) \triangleq BN = \tau_0 + \tau_1 q^{-1} + \cdots + \tau_{\delta\tau} q^{-\delta\tau}$$

$$\gamma(q^{-1}) \triangleq AM = 1 + \gamma_1 q^{-1} + \cdots + \gamma_{\delta\gamma} q^{-\delta\gamma}$$

$$\alpha(q^{-1}) \triangleq \gamma + q^{-1}Q = 1 + \alpha_1 q^{-1} + \cdots + \alpha_{\delta\alpha} q^{-\delta\alpha}. \quad (3.1)$$

*Theorem 1:* Assume the received data to be accurately described by (2.1) and (2.2). The general DFE (2.3) then attains the global minimum of (2.4) if and only if the filters $S/R$ and $Q/P$ have the same coprime factors as

$$\frac{S}{R} = \frac{S_1 N}{M}$$

$$\frac{Q}{P} = \frac{Q}{AM} \quad (3.2)$$

where $S_1$ and $Q$, together with polynomials $L_1$ and $L_2$, satisfy the two coupled polynomial equations

$$\alpha = q^{n-k}\tau S_1 + \gamma L_{1*} \quad (3.3a)$$

$$qL_{2*} = -\rho\gamma_* S_1 + q^{-n+k}\tau_* L_{1*} \quad (3.3b)$$

with polynomial degrees

$$\delta S_1 = \delta L_1 = n-k$$

$$\delta Q = \delta L_2 = \max(\delta\gamma, \delta\tau) - 1. \quad (3.4)$$

The minimal mean-square estimation error is

$$E|z(t)|^2_{\min} = \frac{\lambda_d}{2\pi j}\oint_{|z|=1} L_1 L_{1*} + \rho S_1 S_{1*} \frac{dz}{z}$$

$$= \lambda_d \left( \sum_{j=0}^{n-k} |l_j|^2 + \rho |s_j|^2 \right). \quad (3.5)$$

*Proof:* See Appendix A. □

*Remark:* Note that (3.3) represents two polynomial equations in both $q$ and $q^{-1}$, containing four unknown polynomials $(S_1(q^{-1}), Q(q^{-1}), L_{1*}(q), L_{2*}(q))$. The degrees (3.4) are derived from the constraint that the variables should cover the maximal occurring power of $q^{-1}$ and $q$ in (3.3). With higher degrees, the superfluous coefficients will be zero. With lower degrees, no solution can be found.

An explicit solution to (3.3) is given by the following result.

*Theorem 2:* $S_1$, $\bar{L}_1$ and $Q$ calculated in the following way provide the unique solution to the polynomial equations (3.3).

1) Solve for the coefficients of the polynomials $S_1(q^{-1})$ and $\overline{L}_1(q^{-1})$ in

$$\left[\begin{array}{cccc|cccc} \tau_0 & & & 0 & 1 & & & 0 \\ \vdots & \ddots & & & \gamma_1 & \ddots & & \\ \tau_{n-k} & & \cdots & \tau_0 & \gamma_{n-k} & \cdots & \gamma_1 & 1 \\ \hline \rho & \rho\gamma_1^* & \cdots & \rho\gamma_{n-k}^* & -\tau_0^* & & \cdots & -\tau_{n-k}^* \\ & \ddots & & \rho\gamma_1^* & & \ddots & & \vdots \\ 0 & & & \rho & 0 & & & -\tau_0^* \end{array}\right] \cdot \left[\begin{array}{c} s_0 \\ \vdots \\ s_{n-k} \\ \hline l_{n-k}^* \\ \vdots \\ l_0^* \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]. \tag{3.6}$$

2) With $\{s_i\}$ and $\{l_j^*\}$ obtained from Step 1, perform the multiplication

$$\left[\begin{array}{cccc|cccc} \tau_0 & & & 0 & 1 & & & 0 \\ \vdots & \ddots & & & \gamma_1 & \ddots & & \\ \vdots & & & \tau_0 & \vdots & & & 1 \\ \tau_{\delta\tau} & & & \vdots & \gamma_{\delta\gamma} & & & \gamma_1 \\ & \ddots & & \vdots & & \ddots & & \vdots \\ 0 & & & \tau_{\delta\tau} & 0 & & & \gamma_{\delta\gamma} \end{array}\right] \cdot \left[\begin{array}{c} s_0 \\ \vdots \\ s_{n-k} \\ l_{n-k}^* \\ \vdots \\ l_0^* \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ \alpha_1 \\ \vdots \\ \alpha_{\delta\alpha} \end{array}\right] \tag{3.7}$$

yielding the coefficients of the polynomial $\alpha(q^{-1})$.

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Let $\{h_i\}_0^\infty$ be the impulse response of $\tau(q^{-1})/\gamma(q^{-1})$:

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Equation (3.9) provides a relation between $S_1$, $\overline{L}_1$ and the impulse response of the equalized channel

$$\begin{pmatrix} C_{eq_k} \\ \vdots \\ C_{eq_n} \end{pmatrix} = \begin{pmatrix} h_0 & & 0 \\ \vdots & \ddots & \\ h_{n-k} & \cdots & h_0 \end{pmatrix} \begin{pmatrix} s_0 \\ \vdots \\ s_{n-k} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} l_{n-k}^* \\ \vdots \\ l_0^* \end{pmatrix}. \tag{3.11}$$

Substitution of $\{l_i^*\}$ from (3.11) into the lower blocks of (3.6) results in the following linear system of order $n-k+1$

$$\left(\begin{bmatrix} \rho & \cdots & \rho\gamma_{n-k}^* \\ & \ddots & \vdots \\ 0 & & \rho \end{bmatrix} + \begin{bmatrix} \tau_0^* & \cdots & \tau_{n-k}^* \\ & \ddots & \vdots \\ 0 & & \tau_0^* \end{bmatrix} \cdot \begin{bmatrix} h_0 & & 0 \\ \vdots & \ddots & \\ h_{n-k} & \cdots & h_0 \end{bmatrix}\right) \begin{pmatrix} s_0 \\ \vdots \\ s_{n-k} \end{pmatrix} = \begin{pmatrix} \tau_{n-k}^* \\ \vdots \\ \tau_0^* \end{pmatrix}. \tag{3.12}$$

[1] Factors common to $\tau$ and $\gamma$ must also be factors of $\alpha$, according to (3.3a). From (3.8) it is evident that they then also are factors of $Q$. Thus, they will be common factors in the transfer function $Q/P = Q/\gamma$.

TABLE I
THE DFE OPTIMIZATION ALGORITHM[a]

| | Approximate Number of Multiplications |
|---|---|
| 1) Compute the Markov parameters $\{h_i\}_o^{n-k}$. | $2N\delta Q$ |
| 2) Form the matrix in (3.12). | $\frac{1}{3}N^3+\frac{1}{2}N^2$ |
| 3) Solve (3.12) for $S_1$. | $\frac{1}{3}N^3+\frac{3}{2}N^2$ |
| 4) Using $S_1$, calculate $\overline{L}_1$ from (3.11). | $\frac{1}{2}N^2$ |
| 5) The lower part of (3.7) gives $\alpha$. | $N\delta Q$ |
| 6) $Q$ is obtained from (3.8), and the filters (3.2) are used. | $\frac{2}{3}N^3+\frac{5}{2}N^2+3N\delta Q$ |

[a]With an indication of the computational complexity of each step. The number $N \triangleq n-k+1=\delta S_1+1$ can often be chosen rather small (see e.g., the example in Section VI).

A complete algorithm based on (3.12) is summarized in Table I.

The properties of the optimal DFE are emphasized in some more detail next.

1) It is efficient to whiten the noise. The forward filter $S/R$ contains the *inverse noise description in cascade with a transversal filter* $S_1$ *of order* $n-k$. A consequence of this is that any continuous-time receiver filter that colors the noise will be spectrally eliminated by the DFE forward filter. After noise inversion, we have to equalize a channel $q^{-k}\tau/\gamma = q^{-k}BN/AM$, cf. (3.10). Therefore, the polynomials $S_1$, $Q$, and $P$ are determined exclusively by the polynomials $\tau$ and $\gamma$, not by their separate factors $A$, $B$, $M$, and $N$.
2) A conventional DFE-structure (transversal filters both in the forward and backward links) is optimal if and only if $M=1$, $A=1$. In other words, the channel must be adequately described by a transversal filter, and the noise statistics by an autoregressive process.
3) The solution given in Theorem 1 provides us with an optimal *filter structure* and optimal *polynomial degrees*. Hence, unnecessary overparametrization is avoided. It also gives guidelines on how to choose filter degrees in a conventional DFE-structure. The number of smoothing lags $n$ is a user choice. It may often be chosen rather small (yet $n \geq k$), see the example in Section VI. Usually, $n$ should be chosen greater than or equal to the channel bulk delay, so that the major part of the received impulse energy, caused by $d(t-n)$, can be used by the filter at time $t$.
4) If the impulse response of the channel decays slowly (zeros of $A$ close to the unit circle), the use of $P=\gamma$ will effectively reduce the number of required feedback filter parameters, compared to the use of a transversal filter approximation. In most situations, this reduces the filter complexity. (An exception is when binary real data are used: in a transversal filter, no multiplications are then required.)
5) In the noise-free case ($\rho=0$), the structure of an optimal DFE can be interpreted in the following way: replacing the decision module with unity, a filter that is the inverse of the channel is obtained. This is easily seen from (3.3). With $\rho=0$, we have $L_1=L_2=0$ that gives $\alpha=q^{n-k}\tau S_1$. Regarding the forward and feedback links as a closed loop system, we obtain the transfer function (cf. Fig. 1)
$$\frac{SP}{R(P+q^{-1}Q)}=\frac{S_1NA}{\alpha}=\frac{S_1NA}{q^{n-k}NBS_1}=q^{-(n-k)}\frac{A}{B}.$$
As a consequence, $E|z(t)|^2=0$.
6) In (3.5), the first term $L_1L_{1*}$ is caused by residual intersymbol interference from the first $n-k$ taps of the equalized channel ($\lambda_d\Sigma_{j=1}^{n-k}|l_j|^2$). It is also caused by the deviation of the reference tap (at time index $n-k$) from 1 ($\lambda_d|l_o|^2$), see (3.9). As in all DFE's, the equalized channel impulse response beyond time index $n-k$ is canceled completely by the feedback filter. The tails of past digits thus do not affect the present decision. The second term in (3.5) $S_1S_{1*}$ represents noise transmission.

   Instead of the mse criterion (2.4), *the zero-forcing criterion* [1], [2], has occasionally been used. Intersymbol interference is then eliminated ($L_1=0$), at the price of higher noise transmission, compared to mse optimized forward filters. In our formalism, a zero-forcing equalizer is obtained by setting $L_1=0$ in (3.3). Just solve (3.3a) with $L_1=0$, i.e., $\gamma = q^{n-k}\tau S_1 - q^{-1}Q$, with respect to $S_1(q^{-1})$ and $Q(q^{-1})$. The most relevant criterion would be *minimum probability of error* (MPE), which leads to a nonlinear optimization problem. Monsen [3] has concluded that consideration of MPE and mse lead to essentially the same bit error probability.
7) The denominator polynomials $R$ and $P$ are stable by construction, since $A$ and $M$ are stable. (In an adaptive algorithm, stability of the estimates $\hat{A}$ and $\hat{M}$, or $\hat{R}$ and $\hat{P}$, would be required. This could however easily be handled by stability monitoring in an identification algorithm.)

In decision feedback equalizers, a single incorrect decision can result in a long burst of errors. For many channels, the resulting increase in error probability is small. The effect is severe if feedback filters have long impulse responses; in our case, if $\gamma = AM$ has zeros close to the unit circle. Considerable effort has been spent on deriving bounds on the error probability, including such error bursts [1], [6], [7]. Tradeoffs that reduce error burst lengths have been suggested [9], [10]. Instead of calculating the statistics of error bursts, one may think of ways to eliminate them. One way of doing this is by means of an estimate of the estimation error $\hat{z}(t-n) \triangleq \tilde{d}(t-n) - \hat{d}(t-n)$, cf. Fig. 1. Assume that a delay between a preliminary *soft* decision, and a final *hard* decision is acceptable. Furthermore, assume PAM binary data ($m=2$) and a low noise level. (Large $m$ and/or high noise levels would reduce the effectivity of the scheme.) Normally, $|\hat{z}|$ will then be small, compared to $|d|$. If $|\hat{z}|$ suddenly becomes large on several consecutive samples, a burst of errors has probably started. If so, reset the states of the filters, change the value of the bit at the beginning of the suspected error burst and repeat the filtering. Recently, Dahlman and Gudmundsson [11] tested a simple error burst supression algorithm by means of simulations. They found that the error bursts were reduced substantially.

## IV. An Optimal Linear Feedback Equalizer

In this section, we will discuss linear recursive equalizers, also called linear feedback equalizers (LFE). This means that only the forward filter is used in (2.3). Linear estimators of $d(t)$, which minimize the MSE criterion, are also known as *deconvolution filters* or *Kalman equalizers*. Equalizers for FIR channels with white noise are well known [8], [17], [18].

In [12] a new linear deconvolution filter was derived for real-valued signals. It minimizes the MSE criterion with respect to $(S, R)$ for the general channel structure described in (2.1). It can also handle correlated input sequences $d(t)$. For digital data, the output from this filter could be entered into a decision module.

For a white data sequence, the optimal filter is given by

$$\hat{d}(t|t+n) = \frac{S(q^{-1})}{R(q^{-1})} y(t+n)$$
$$= q^n \frac{S_1(q^{-1}) N(q^{-1}) A(q^{-1})}{\beta(q^{-1})} y(t) \quad (4.1)$$

where $\beta(q^{-1})$ is the stable monic solution to a *spectral factorization* equation

$$r\beta\beta_* = \tau\tau_* + \rho\gamma\gamma_* \qquad (r \text{ is a scalar}) \qquad (4.2)$$

and $S_1(q^{-1})$, together with a polynomial $L_*(q)$, is the solution of the linear polynomial equation

$$q^{-n+k}\tau_* = r\beta_* S_1 + qL_* \qquad (4.3)$$

with polynomial degrees

$$\delta S_1 = n - k, \qquad \delta L = \max(\delta\tau - n + k, \delta\beta) - 1. \quad (4.4)$$

The minimal estimation error is given by

$$E|z(t)|^2_{\min} = \frac{\lambda_d}{2\pi j} \oint_{|z|=1} \frac{LL_* + \rho\gamma\gamma_*}{r\beta\beta_*} \frac{dz}{z}. \quad (4.5)$$

Readers interested in the derivations of these results are referred to [12], [13].

The linear equalizer may be compared to the DFE from the previous section.

- Linear equalizers are a special case of the general DFE-structure ($Q = 0$). Thus, a correctly tuned general DFE is never inferior to a linear equalizer, if correct past decisions can be assumed. In the limit $n \to \infty$, it can be shown that their MSE is equal only if there is no intersymbol interference [4].
- The LFE consists of a *signal whitening filter* $AN/\beta$ (the inverse of an innovations model of $y(t)$) in cascade with a FIR-filter $S_1$. As forward part of the DFE, we have a *noise whitening filter* $N/M$ in cascade with a FIR-filter. Computation of a spectral factor $\beta$ is not needed for designing the DFE.
- If the channel is low-pass, the LFE will be a *high-pass filter*. Note, in particular, that $A(q^{-1})$ is a numerator factor in (4.1). For channels with deep in-band nulls, the LFE-transfer function has strong resonances, if $\rho$ is small. Consequently, noise is amplified. This explains the well-known unsatisfactory performance of linear equalizers in many applications [15]. With decision feedback, the forward filter needs not to be an inverse filter. Instead, the inverse effect is handled by the feedback loop. This results in a lower noise amplification. A small reduction of the mse means a large reduction of the bit error rate for Gaussian distributed noise. The consequence is often a drastically reduced error probability in DFE's, compared to LFE's.

## V. The Asymptotic Structure of Equalizers with Large Smoothing Lags

Some asymptotic expressions, valid when the smoothing lag $n \to \infty$, will be derived. Such nonrealizable filters correspond to noncausal Wiener filters. The expressions are of interest when comparing our solution to the noncausal filters derived in, for example, [2]–[5].

*Theorem 4:* When $n \to \infty$, the linear equalizer (4.1), $\hat{d}(t) = (S/R)y(t+n)$, approaches the following filter

$$\left(\frac{S}{R}\right)^{\infty}_{\text{LFE}} \triangleq \lim_{n\to\infty} q^n \frac{S}{R}$$
$$= q^k \frac{B_* N_* NA}{BB_* NN_* + \rho AA_* MM_*} \quad (5.1a)$$
$$= q^k \frac{B_*}{A_*} \frac{1}{\left(\dfrac{BB_*}{AA_*} + \rho \dfrac{MM_*}{NN_*}\right)}. \quad (5.1b)$$

*Proof:* The expressions constitute a noncausal Wiener filter. They can be derived easily from the standard Wiener filter formula $\Phi_{dy}/\Phi_{yy}$, where $\Phi_{dy}$ is the cross spectral density between $d$ and $y$, and $\Phi_{yy}$ is the spectral density of $y$.

As an alternative proof, it can be noted, in the same way as in Appendix D, that $L_*(q) \to 0$ as $n \to \infty$. Equation (4.3) then reduces to $q^k\tau_* = r\beta_*(q^n S_1)_{n\to\infty}$. The impulse response of $q^n S_1(q^{-1})$ thus approaches that of $q^k\tau_*(q)/r\beta_*(q)$ as $n \to \infty$. Substitution of this expression into (4.1) and the use of (4.2) gives (5.1a). □

*Theorem 5:* When $n \to \infty$, the forward part of the general decision feedback equalizer approaches the following filter

$$\left(\frac{S}{R}\right)^{\infty}_{DFE} \triangleq \lim_{n\to\infty} q^n \frac{S}{R}$$

$$= \left(\frac{S}{R}\right)^{\infty}_{LFE}\left(1+q^{-1}\frac{Q}{P}\right) \tag{5.2a}$$

$$= q^k \frac{N}{M}\frac{\tau_*\alpha}{r\beta\beta_*} = q^k \frac{N}{M}\frac{\tau_*}{r\beta_*} \tag{5.2b}$$

where $Q/P = Q/AM$ is the feedback filter, $\alpha$ is defined by (3.1), $r\beta\beta_*$ by (4.2) and $(S/R)_{LFE}$ is the asymptotic linear equalizer discussed in Theorem 4.

*Proof:* See Appendix D. □

The connection (5.2a) between the asymptotic expressions for the DFE forward filter and the linear equalizer has been derived previously for continuous time channel models [2], [5]. Note that this connection is an approximate one for realizable filters. It does *not* hold exactly for $n<\infty$. In the limit $n\to\infty$, $\alpha(q^{-1})=\beta(q^{-1})$. Thus, the limiting expression of the DFE feedback polynomial $Q(q^{-1})$ is, cf. (3.8), $Q(q^{-1})=q(\beta(q^{-1})-\gamma(q^{-1}))$.

The expression $q^kB_*(q)/A_*(q)$ in (5.1b) is called a *matched filter*. It represents a noncausal impulse response, time reversed with respect to that of a received isolated single pulse $(q^{-k}B(q^{-1})/A(q^{-1}))d(t)$, It is, of course, nonrealizable when $A(q^{-1})\neq 1$.

Many workers, who have discussed optimization of equalizers, have used structures that contain continuous-time filters matched to the received pulse [2]–[5], [15]. The expressions (5.1b) and (5.2a) confirm that optimal linear equalizers and DFE forward filters can be expressed in this way, in the limit $n\to\infty$. These formulas contain the product of a filter matched to an equivalent discrete-time channel model and an additional filter. There do, however, exist alternative expressions, which do not contain any matched filter, as is indicated by (5.1a) and (5.2b).

The structure with a matched filter in cascade with another filter is optimal *only* in the limit $n\to\infty$. The expressions (3.2)–(3.3) and (4.1)–(4.3) for the calculation of realizable DFE's and LFE's do not contain any "matched filters." This is not surprising, since the constraint of realizability (stability and a finite smoothing lag $n$) excludes such expressions from the filter structure.

## VI. A Numerical Example

Consider the following channel and noise description, with real coefficients:

$$B(q^{-1}) = 0.407+0.815q^{-1}+0.407q^{-2},$$

$$M(q^{-1}) = 1-0.8q^{-1}$$

$$A(q^{-1}) = N(q^{-1}) = 1, \qquad Ed(t)^2=1, \qquad Ev(t)^2=\rho.$$

The disturbance $v(t)$ is zero-mean and Gaussian. The data sequence consists of real antipodal binary pulses ($m=2$). $B(q^{-1})$ is a well-known channel model given, for example, in [15]. We also assume a moderately colored noise described by $M(q^{-1})$. In Fig. 3, we compare the probability of error for the linear feedback equalizer (LFE), the general decision feedback equalizer (GDFE) introduced in Section III and a conventional decision feedback equalizer (DFE). For the DFE, the transversal filters have degrees determined from the condition (3.4) as if the noise were white. For comparison, the result when using the DFE's on the channel with white noise is also shown.[2]

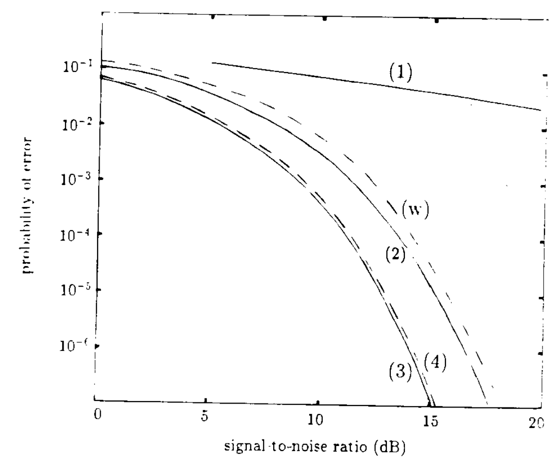


Fig. 3. Bit-error rates as function of signal-to-noise ratio for different equalizers. SNR is defined as $Es(t)^2/Ef(t)^2$. ($w$) GDFE and DFE when the additive noise is white. (1) LFE, $n=10$ (12 forward parameters), (2) DFE, $n=10$ (11 forward, 2 feedback taps), (3) GDFE, $n=10$ (12 forward, 3 feedback parameters), (4) GDFE, $n=2$ (4 forward, 3 feedback parameters).

From Fig. 3, it is apparent that if a colored noise is present, it is advantageous to take this information into account. The general DFE does this in a superior way, compared to the conventional DFE. By increasing the number of taps, the impulse responses of the transversal filters in the DFE could be made to approach those of the recursive filters in the GDFE. The price to be paid for this would, of course, be a higher number of parameters.

The optimal LFE will, in this case, not be able to compete at all. This is often the case, as has been noted in, for example, [2], [15], [23]. Since the zeros are close to the unit circle, the LFE performs poorly even if large smoothing lags $n\geq 10$ are used. For the GDFE however, comparison of (3), $n=10$, with (4), $n=2$, indicates that a good performance can be achieved with a very small number of filter parameters. Choosing $n$ of the same order as the channel bulk delay seems appropriate. (If the

[2]The bit error for DFE's in Fig. 3 was calculated under the assumption of correct past decisions, i.e., no error bursts. For a given bias, due to intersymbol interference in the imperfectly equalized channel, the probability of error due to the Gaussian noise was calculated. Summation over all possible bit patterns in the equivalent equalized channel transfer function (3.9) gave the bit-error rate. (The curve corresponding to ($w$) in [15, p. 386], obtained by simulation, gives a somewhat lower error rate than ours, which is obtained by direct calculation.)

channel impulse response has a long significant tail, $n$ should be chosen larger.)

Let us finally illustrate the calculations in Theorem 2 in the case of $n = 1$ (one lag smoothing) and $\rho = 0.061$. With $\tau = B$ and $\gamma = M$, (3.6) becomes

$$\left(\begin{array}{cc|cc} 0.407 & 0 & 1 & 0 \\ 0.815 & 0.407 & -0.8 & 1 \\ \hline 0.061 & -0.0488 & -0.407 & -0.815 \\ 0 & 0.061 & 0 & -0.407 \end{array}\right) \left(\begin{array}{c} s_0 \\ s_1 \\ \hline l_1 \\ l_0 \end{array}\right) = \left(\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 0 \end{array}\right)$$

giving

$$S_1(q^{-1}) = 0.5322 + 0.7056q^{-1}$$
$$L_1(q^{-1}) = 0.1058 - 0.2166q^{-1}.$$

(Alternatively, (3.12) and (3.11) could have been used.) The multiplication (3.7) results in $\alpha(q^{-1}) = 1 + 0.7071q^{-1} + 0.2872q^{-2}$ and (3.8) gives

$$Q(q^{-1}) = q(\alpha(q^{-1}) - M(q^{-1})) = 1.5071 + 0.2872q^{-1}.$$

Thus, with $N = 1$ and $A = 1$, the optimal general DFE is given by

$$\hat{d}(t-1|t) = \frac{(0.5322 + 0.7056q^{-1})}{1 - 0.8q^{-1}} y(t) - \frac{(1.5071 + 0.2872q^{-1})}{1 - 0.8q^{-1}} \tilde{d}(t-2).$$

The minimal mse becomes 0.106. The equalized channel (from $d(t)$ to $\hat{d}(t-1|t)$) is, cf. (3.9):

$$q^{-1} - \bar{L}_1(q^{-1}) = 0.2166 + 0.8942q^{-1}.$$

## VII. Conclusions

An explicit solution to the problem of optimizing decision feedback equalizers has been discussed. An approach based on polynomial equations has been introduced. For a channel with infinite impulse response and colored measurement noise, mse-optimal realizable (stable and finite smoothing lag) filters were presented. Correct past decisions were assumed. The structure and degree of the optimal forward and feedback filters are evident from the solution. In general, the filters are of recursive (IIR)-type. For transmission channels with finite impulse response and autoregressive noise, the minimum mean square error can, however, be attained with transversal feedback and forward filters. When the noise is nonwhite, it is optimal to include a noise-whitening filter (the inverse noise model) in cascade with a transversal filter as the forward part.

Optimal filters can be calculated in a very simple way, essentially by solving a system of linear equations $Ax = B$, where $A$ contains transfer function coefficients from the channel and noise models. A simple expression for the minimal mse has also been presented. The DFE has been compared to the optimal linear recursive equalizer discussed more extensively in [12].

The optimal DFE does not contain any nonrealizable "matched filter." Optimization under the constraint of realizability excludes the presence of matched filters for finite smoothing lags $n$. We have discussed the filter design exclusively in discrete time. The combined optimization of continuous-time receiver filters (before sampling) and the discrete-time parts of the DFE is a problem for further research. (We do not believe that continuous-time matched filters will be part of such designs, if realizability is required.)

The presented solution is, by itself, a tool for theoretical investigation. Research is currently under way to investigate if it can also be used as the central part of an adaptive equalizer. Such an algorithm would require the estimation of $A$, $B$, $M$, and $N$ from output data $\{y(t)\}$ and known training sequences $\{d(t)\}$. Between the training periods, the channel parameters may need to be tracked. A successful adaptive deconvolution algorithm has been developed along these lines [29]. The identifiability properties of channel and noise models from output data only have been investigated [30]. The need to use higher-order statistics to track nonminimum phase channels is another issue. (Interestingly, for typical mobile radio channels, it may be possible to track the channel variations without using higher-order statistics. This is under current investigation.) An indirect, channel-model based, adaptation would eliminate the problem of estimator divergence due to catastrophic error propagation. This is a risk, when decision directed adaption is used in DFE's. It would also reduce the number of parameters that need to be adapted; the number of filter parameters is mostly larger than the number of channel and noise model parameters.

## Appendix

### A. Proof of Theorem 1

The expressions (3.2) and (3.3) were originally derived by differentiating the criterion (2.4) with respect to the coefficients of $S$, $R$, $Q$, and $P$. Here, it will just be verified that (3.2) and (3.3) must be satisfied by an optimal solution. First, we show that (3.2) and (3.3) imply an estimation error given by (3.5). Secondly, it is shown that (3.5) is the minimal value, and that it is attained only when (3.2) and (3.3) are satisfied.

Using (2.1), (2.3), and (2.4), the estimation error becomes (cf. Fig. 2)

$$z(t) = \frac{(\mathrm{ARPN} - q^{n-k}\mathrm{BPSN} + q^{-1}\mathrm{QARN})d(t) - q^{n}\mathrm{SMA}v(t)}{\mathrm{ARPN}}. \tag{A.1}$$

With the filters (3.2), possible unstable factors of $N$ will be canceled by $S = S_1N$. Since $A$, $R$, and $P$ are stable, $z(t)$ will thus be stationary and Parseval's formula can be used to express its variance.

Consider two complex-valued signals $x(t) = G(q^{-1})e(t)$ and $w(t) = H(q^{-1})e(t) = \sum_0^\infty h_j e(t-j)$, where $e(t)$ is white noise. If we write Parsevals relation for complex signals [31] in the stochastic case, a cross correlation can, in our notation, be expressed as

$$\begin{aligned} &Ex(t)w(t)^* \\ &= \frac{\lambda_e}{2\pi j}\oint_{|z|=1} G(z^{-1})H^*(z^*)\frac{dz}{z} \\ &= \frac{\lambda}{2\pi j}\oint_{|z|=1} G(z^{-1})\left(h_0 + h_1 z^* + h_2 z^{*2} + \cdots\right)^* \frac{dz}{z} \\ &= \frac{\lambda_e}{2\pi j}\oint_{|z|=1} G(z^{-1})\left(h_0^* + h_1^* z + h_2^* z^2 + \cdots\right)\frac{dz}{z} \\ &= \frac{\lambda_e}{2\pi j}\oint_{|z|=1} G(z^{-1})H_*(z)\frac{dz}{z}. \end{aligned}$$

This is why the conjugate polynomials $P_*$, defined in the beginning of Section III, are of use.

We now use this relation, and the assumption that $d(t)$ and $v(t)$ are zero-mean and mutually independent. With (2.2) and $S = S_1 N, R = M, P = AM$ from (3.2), we get

$$E|z(t)|^2 = \frac{\lambda_d}{2\pi j}\oint_{|z|=1} \frac{\left(\gamma - z^{n-k}\tau S_1 + z^{-1}Q\right)\left(\gamma_* - z^{k-n}\tau_* S_{1*} + zQ_*\right)}{\gamma\gamma_*}\frac{dz}{z} + \frac{\lambda_d}{2\pi_j}\oint \rho S_1 S_{1*}\frac{dz}{z}.$$

The use of (3.3a) transforms this expression into (3.5). Next, it will be shown that (3.5) is the minimal value.

Let an arbitrary estimate of $d(t)$, assuming correct past decisions, be written

$$\hat{d}(t|t+n) = \frac{S}{R}y(t+n) - q^{-1}\frac{Q}{P}d(t) + w(t) \qquad \text{(A.2)}$$

where $(S, R, Q, P)$ are determined as in Theorem 1 and $w(t)$ is an arbitrary additional signal. It is allowed to be a sum of linear combinations of received signals $y$ up to time $t+n$ and correct past decisions $\hat{d} = d$ up to time $t-1$. It will be demonstrated that $w(t) = 0$ is the optimal choice.

Let $z(t)$ be the estimation error when Theorem 1 is satisfied ($w(t) = 0$). Using (A.2), the estimation error variance is then given by

$$\begin{aligned} &\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} E|d(t) - \hat{d}(t|t+n)|^2 \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} E|z(t)|^2 - Ez(t)w(t)^* \\ &\quad - Ew(t)2t^* + E|w(t)|^2. \qquad \text{(A.3)} \end{aligned}$$

If $w(t)$ is nonstationary, the ensemble means would change with time, and the criterion could be undefined. Assume $w(t)$ to be stationary. It can then be expressed as

$$w(t) = q^n\frac{G(q^{-1})}{H(q^{-1})}y(t) - q^{-1}\frac{F(q^{-1})}{T(q^{-1})}d(t) \qquad \text{(A.4)}$$

where $H$ and $T$ are restricted to be stable and $G$ cancels possible unstable factors of $N$, so that the transfer function from $v(t)$ to $w(t)$ is stable.

We now demonstrate that the mixed terms in (A.3) vanish, Due to symmetry, it is sufficient to consider $Ez(t)w^*(t)$. With the aid of (3.2), (3.3), (A.1), and (A.4), it can be expressed as

$$\begin{aligned} 2Ez(t)w(t)^* = 2E&\left[\left(1 - q^{n-k}\frac{\tau S_1}{\gamma} + q^{-1}\frac{Q}{\gamma}\right)d(t) - q^n S_1 v(t)\right] \\ &\cdot\left[q^n\frac{G}{H}\left(q^{-k}\frac{B}{A}d(t) + \frac{M}{N}v(t)\right) - q^{-1}\frac{F}{T}d(t)\right]^* \\ = 2E&\left[L_{1*}d(t)\left\{\left(q^{n-k}\frac{BG}{AH} - q^{-1}\frac{F}{T}\right)d(t)\right\}^*\right. \\ &\left.\cdot - q^n S_1 v(t)\left\{q^n\frac{MG}{NH}v(t)\right\}^*\right]. \end{aligned}$$

In the first equality, a slightly rewritten (A.1) was used, and in the last equality equation (3.3a) was applied. There, $L_{1*}(q)d(t)$ represents $q^{-n+k}L_{1*}q^{n-k}d(t) = \overline{L}_1(q^{-1})d(t+n-k)$. Parsevals formula, with $z$ exchanged for $q$, gives

$$\begin{aligned} &Ez(t)w(t)^* \\ &= 2\frac{\lambda_d}{\pi j}\oint_{|z|=1}\left(L_{1*}\frac{z^{-n+k}\tau_* G_*}{A_* N_* H_*} - L_{1*}z\frac{F_*}{T_*} - \rho S_1\frac{\gamma_* G_*}{A_* N_* H_*}\right)\frac{dz}{z} \\ &= 2\frac{\lambda_d}{\pi j}\oint_{|z|=1}\left(\frac{\left(z^{-n+k}\tau_* L_{1*} - \rho\gamma_* S_1\right)}{A_* N_*}\frac{G_*}{H_*} - zL_{1*}\frac{F_*}{T_*}\right)\frac{dz}{z} \\ &= 2\frac{\lambda_d}{\pi j}\oint_{|z|=1}\left(\frac{L_{2*}}{A_* N_*}\frac{G_*}{H_*} - L_{1*}\frac{F_*}{T_*}\right)dz = 0. \qquad \text{(A.5)} \end{aligned}$$

In the third equality, equation (3.3b) was used. The polynomials $A(z^{-1})$, $H(z^{-1})$ and $T(z^{-1})$ are assumed stable. They have all zeros inside the unit circle. Thus, $A_*(z)$, $H_*(z)$ and $T_*(z)$ have all zeros *outside* the unit circle. The same is true for the poles of $G_*(z)/N_*(z)$, since $G$ is required to cancel all unstable factors of $N$. Consequently, the integrand in (A.5) has no poles inside the unit circle, so the integral will vanish.

With $z(t)$ and $w(t)$ stationary and $Ez(t)w(t)^* = 0$, the estimation error (A.3) becomes

$$E|d(t) - \hat{d}(t|t+n)|^2 = E|z(t)|^2 + E|w(t)|^2. \qquad \text{(A.6)}$$

Evidently, it is minimized by choosing $w(t) = 0$, since $E|w(t)|^2$ is nonnegative. Any DFE that does not satisfy (3.2) and (3.3) would, by definition, correspond to $w(t) \neq 0$ in (A.2). No such equalizer attains the minimal mse. □

### B. Proof of Theorem 2

Multiply (3.3a) by $q^{-n+k}$ and (3.3b) by $-q^{-\delta L_2 - 1} = -q^{-\max\{\delta\tau,\delta\gamma\}}$, to obtain equations in powers of $q^{-1}$ only. This give

$$\begin{aligned} q^{-n+k}\alpha &= \tau S_1 + \gamma\overline{L}_1 \\ -\overline{L}_2 &= q^{-r_1}\rho\bar{\gamma}S_1 - q^{-r_2}\bar{\tau}\overline{L}_1 \end{aligned} \qquad \text{(B.1)}$$

where $r_1 = \max\{\delta\tau, \delta\gamma\} - \delta\gamma$, and $r_2 = \max\{\delta\tau, \delta\gamma\} - \delta\tau$.

Rewrite (B.1) in matrix form

$$\left[\begin{array}{ccc|ccc} \tau_0 & & 0 & 1 & & 0 \\ \vdots & \ddots & & \gamma_1 & \ddots & \\ \tau_{\delta\tau} & & \tau_0 & \vdots & & 1 \\ & \ddots & \vdots & \gamma_{\delta_\gamma} & & \gamma_1 \\ & & \tau_{\delta_\tau} & & \ddots & \vdots \\ 0 & & & 0 & & \tau_{\delta_\gamma} \end{array}\right] \begin{bmatrix} s_0 \\ \vdots \\ s_{n-k} \\ l^*_{n-k} \\ \vdots \\ l^*_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \alpha_1 \\ \vdots \\ \alpha_{\delta_\alpha} \end{bmatrix} \tag{B.2}$$

$$\left[\begin{array}{ccc|ccc} 0 \updownarrow r_1 & & & 0 \updownarrow r_2 & & \\ \rho\gamma^*_{\delta_\gamma} & & 0 & -\tau^*_{\delta_\tau} & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ \rho\gamma^*_1 & & \rho\gamma^*_{\delta_\gamma} & \vdots & & -\tau^*_{\delta_\tau} \\ \rho & & \vdots & -\tau^*_0 & & \vdots \\ & \ddots & \rho\gamma^*_1 & & \ddots & \vdots \\ 0 & & \rho & 0 & & -\tau^*_0 \end{array}\right] \begin{bmatrix} s_0 \\ \vdots \\ s_{n-k} \\ l^*_{n-k} \\ \vdots \\ l^*_0 \end{bmatrix}$$

$$= \begin{bmatrix} -l^*_{2\delta L_2} \\ \vdots \\ -l^*_{20} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{B.3}$$

A problem with (B.2) and (B.3) is that their right-hand sides contain unknowns. The vector $(s_0, \cdots, s_{n-k}, l^*_{n-k}, \cdots, l^*_0)$ can, however, be calculated from a subset of equations with known right-hand side. The combination of the first $n-k+1$ equations from (B.2) with the last $n-k+1$ equations from (B.3) is adequate for this purpose. Hence, we get (3.6). The solution to (3.6) is used in (B.2) to compute $\alpha(q^{-1})$. (It could also be inserted into (B.3) to determine $L_2(q^{-1})$.) Using $\alpha(q^{-1})$ and the known $\gamma(q^{-1}) = A(q^{-1})M(q^{-1})$, $Q(q^{-1})$ is determined from (3.1). Note that the leading coefficients of the monic polynomials $\alpha(q^{-1})$ and $\gamma(q^{-1})$ cancel. Thus, $Q(q^{-1})$ is causal.

To prove (3.9), recall Fig. 2. Using (3.1), (3.2), and (3.3a), the transfer function $C_{eq}$, from $d(t)$ to $\hat{d}(t-n|t)$, is found to be

$$\begin{aligned} C_{eq} &= q^{-n}\left(q^{n-k}\frac{\tau S_1(q^{-1})}{\gamma(q^{-1})} - q^{-1}\frac{Q(q^{-1})}{\gamma(q^{-1})}\right) \\ &= q^{-n}(1 - L_{1*}(q)) \\ &= q^{-n} - q^{-k}\overline{L}_1(q^{-1}). \end{aligned}$$

□

## C. Proof of Theorem 3

Proceeding from (3.6), define

$$T \triangleq \begin{pmatrix} \tau_0 & & 0 \\ \vdots & \ddots & \\ \tau_{n-k} & \cdots & \tau_0 \end{pmatrix} \qquad G \triangleq \begin{pmatrix} 1 & & 0 \\ \gamma_1 & \ddots & \\ \vdots & \ddots & \\ \gamma_{n-k} & \cdots \quad \gamma_1 & 1 \end{pmatrix}$$

$$b \triangleq (0, \cdots, 0, 1)^T$$

$$\theta \triangleq (s_0, \cdots, s_{n-k}, l^*_{n-k}, \cdots, l^*_0)^T.$$

Thus, (3.6) may be written

$$\underbrace{\begin{pmatrix} T & G \\ \rho G^H & -T^H \end{pmatrix}}_{A} \theta = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

where $H$ denotes the Hermitian transpose. We obtain

$$\begin{aligned} \det A &= \det\left(-T^H - \rho G^H T^{-1} G\right) \det T \\ &= -\det\left(T^H T + \rho G^H T^{-1} G T\right) \\ &= -\det\left(T^H T + \rho G^H G\right). \end{aligned}$$

In the last equality, we have used the fact that the Toeplitz matrices $T$ and $G$ commute. Since $G^H G$ is always positive definite, and $T^H T$ is positive definite whenever $b_0 = \tau_0 \neq 0$, $\rho G^H G + T^H T$ will be positive definite since $\rho \geq 0$. Thus, it follows that $\rho G^H G + T^H T$ is nonsingular. Consequently, $\det A \neq 0$ and (3.6) will always have a unique solution. □

## D. Proof of Theorem 5

First, note that as $n \to \infty$, less and less information about $d(t-n)$ is contained in the present measurement $y(t)$ [24]. Thus, the leading coefficients of $S_1(q^{-1})$ tend to zero as $n$ increases. This will be true in particular for the first $\delta L_2 + 1$ coefficients. (Note that $\delta L_2 + 1 = \max(\delta\gamma, \delta\tau)$ does not increase as $n$ increases.)

Consider the first $\delta L_2 + 1$ equations of (B.2). Since $\{s_0, \cdots, s_{\delta L_2}\} \to 0$ as $n \to \infty$, it follows that $\{l^*_{n-k}, \cdots, l^*_{n-k-\delta L_2}\} \to 0$, since the right-hand sides of the equations are zero. Thus, the leading coefficients of $\overline{L}_1(q^{-1})$ (which are the leading coefficients of the equalized channel impulse response according to (3.9)) vanish as $n \to \infty$.

Now, consider the polynomial $\overline{L}_2$ in (B.1). According to the first $\delta L_2 + 1$ equations of (B.3), the coefficients of $\overline{L}_2$ are a linear combination of $(s_0, s_1, \cdots, s_{\delta L_2})$ and of $(l^*_{n-k}, l^*_{n-k-1}, \cdots, l^*_{n-k-\delta L_2})$. Thus, $\overline{L}_2(q^{-1})$ goes to zero as $n \to \infty$. Consequently, an asymptotic expression for the DFE feedforward filter can be derived by letting $\overline{L}_2 \to 0$ in (B.1).

Multiplying (3.3b) by $\gamma$ yields

$$qL_{2*}\gamma = -\rho\gamma\gamma_* S_1 + q^{-n+k}\tau_*\gamma L_{1*}. \tag{D.1}$$

Next, multiply (3.3a) by $q^{-n+k}\tau_*$ to get

$$q^{-n+k}\tau_*\alpha = \tau\tau_* S_1 + q^{-n+k}\tau_*\gamma L_{1*}. \tag{D.2}$$

By subtracting (D.1) from (D.2), and considering the limit $n \to \infty$, which implies that $L_{2*}$ vanishes, the following asymptotic expression for the impulse response of $q^n S_1(q^{-1})$ is obtained

$$\lim_{n \to \infty} q^n S_1(q^{-1}) = q^k \frac{\tau_*(q)\alpha(q^{-1})}{r\beta(q^{-1})\beta_*(q)}. \tag{D.3}$$

The use of (D.3) in (3.2) gives

$$\left(\frac{S}{R}\right)^{\infty}_{DFE} \triangleq \lim_{n \to \infty} q^n \frac{S}{R} = q^k \frac{N}{M}\frac{\tau_*\alpha}{\tau\beta\beta_*}. \tag{D.4}$$

On the other hand, by using (5.1a), (4.2) and the definition $\alpha = AM + q^{-1}Q$, it is found that

$$\begin{aligned} \left(\frac{S}{R}\right)^{\infty}_{LFE}\left(1 + q^{-1}\frac{Q}{P}\right) &= \left(\frac{S}{R}\right)^{\infty}_{LFE}\frac{(AM + q^{-1}Q)}{AM} \\ &= q^k\frac{\tau_* NA}{r\beta\beta_*}\frac{\alpha}{AM} = q^k\frac{N}{M}\frac{\tau_*\alpha}{r\beta\beta_*}. \end{aligned} \tag{D.5}$$

The equalities (D.4) and (D.5) together constitute the first two equations in (5.2). Finally, write (D.3) as

$$\frac{r\beta_*(q)}{\tau_*(q)} \lim_{n\to\infty} q^{n-k} S_1(q^{-1}) = \frac{\alpha(q^{-1})}{\beta(q^{-1})}. \qquad \text{(D.6)}$$

Since the left-hand side of (D.6) is anticausal (it contains only positive powers of $q$), while the right-hand side is a causal transfer function, the equality can hold only if $\alpha(q^{-1}) = \beta(q^{-1})$. Hence, we have the last equality in (5.2b). □

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