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Brief Paper

LQG-optimal Feedforward Regulators*†

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Key Words—Feedforward control; discrete time systems; disturbance rejection; linear optimal regulator.

Abstract—A polynomial LQG approach to the design of feedforward regulators is presented. Given a linear system and possibly a prespecified feedback, optimal feedforward filters for non-minimum phase systems can be calculated in a simple way. An infinite horizon criterion including a filtered input signal is minimized. This makes it possible to include frequency-dependent trade-offs between input energy and disturbance rejection in the design. The achievable feedforward control performance turns out to be unaffected by the choice of feedback, if the optimal regulator structure is used. This suggests a simple way of optimizing combined feedback and feedforward regulators: a main output feedback is first optimized with respect to unmeasurable disturbances. The feedforward link is then optimized with respect to the measurable disturbance.

1. *Introduction*

When disturbances can be measured, their influence may be cancelled. From single input linear models, it is simple to compute feedforward filters giving perfect cancellation, if such filters exist. When systems have unstable inverses or significant time dealys, perfect cancellation may, however, be impossible. In other situations, it requires unrealistically large input signals. While the design of feedforward regulators is not trivial in such cases, it is still worthwhile: if measurable disturbances are present, the control performance can often be improved radically by utilizing this information, compared with the use of output feedback only. The reason is that the regulator can begin to act on a disturbance before it has affected the output. Application of feedforward also facilitates the design of stabilizing feedback regulators. With a feedforward link compensating for most of the disturbance, designers can concentrate on the robust stability, rather than the disturbance rejection, of the feedback system.

A systematic method is needed for designing combined feedback–feedforward regulators. Davison (1973, 1976) described methods to cancel measurable deterministic disturbances asymptotically. In the present paper, stochastic disturbances are considered. We will optimize an infinite horizon criterion using a *polynomial LQG* technique. It is suitable both for off-line design and as a basis for adaptive control.

Since a large majority of practical regulator design problems can be decomposed into single input problems, we will discuss scalar systems. Because computer control now dominates in process control applications, discrete time regulators are discussed. The method provides optimal feedforward regulators for non-minimum phase systems. This is important, since sampling often leads to sampled systems with unstable inverses even for continuous-time systems with stable inverses, cf. Åström *et al.* (1984).

Normally, feedforward has to be combined with output feedback in a control system. A new contribution in the present work, presented in Section 3, is the design of optimal feedforward filters used in combination with any prespecified feedback. Pole placement feedback, PID controllers and use of feedforward only are some examples. We also discuss how to combine feedback and feedforward filters. The achievable feedforward control performance turns out to be unaffected by the choice of feedback. This is, however, true only if feedback and feedforward filters are combined in the correct way.

If unmeasureable stochastic disturbances are present, combined feedback and feedforward regulators may be optimized using a simple step-by-step procedure: a main output feedback is first optimized with respect to unmeasurable disturbances. The feedforward link is then optimized with respect to measurable disturbances. This design procedure has been derived by Sternad (1985) and, independently, by Grimble (1986). The result is discussed in Section 4.

2. *Problem formulation*

The true system is assumed to be described by a linear model with the following structure:

$$A(q^{-1})y(t) = q^{-k}B(q^{-1})u(t) + q^{-d}D(q^{-1})w(t) + Ce(t)$$
$$w(t) = \frac{G(q^{-1})}{H(q^{-1})}v(t). \tag{2.1}$$

The main output $y(t)$ and the auxiliary output (measurable disturbance) $w(t)$ are assumed to be measurable without additional measurement noise. $u(t)$ is the input and $e(t)$ is an unmeasurable disturbance. A, $B, \ldots,$ are polynomials in the backward shift operator q^{-1} with degrees na, $nb, \ldots$ The time delays $k > 0$ and $d \geq 0$ may be such that $k > d$ [in which case perfect cancellation of $w(t)$ is impossible]. A and B are not allowed to have unstable common factors. The polynomials A, C, H and G are monic (their leading coefficient is 1). The disturbances are represented by equivalent stochastic models, with $e(t)$ and $v(t)$ being uncorrelated white stationary random sequences. They have zero means and variances Λ_e and Λ_v, respectively. The polynomials C, G and H are stable.

The regulator structure

$$R(q^{-1})u(t) = -\frac{Q(q^{-1})}{P(q^{-1})}w(t) - S(q^{-1})y(t) \tag{2.2}$$

will be used, see Fig. 1. The polynomials R and P are monic and P is required to be stable. This choice of regulator structure will be motivated in Section 3. The corresponding parameter vector is given by

$$\theta = (p_1, \ldots, p_{np}, Q_0, \ldots, Q_{nQ}, r_1, \ldots, r_{nr}, s_0, \ldots, s_{ns})^{\mathrm{T}}. \tag{2.3}$$

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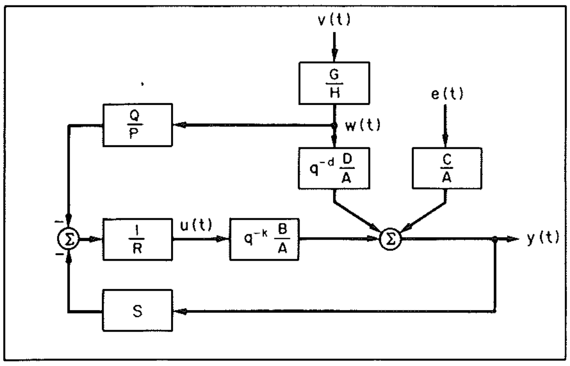

FIG. 1. The system and regulator structure.

The infinite horizon criterion

$$J = \lim_{N\to\infty} \frac{1}{2N} \sum_{t=0}^{N} Ey(t)^2 + \rho E(\tilde{\Delta}(q^{-1})u(t))^2 \qquad (2.4)$$

is to be minimized. The input penalty ρ and the polynomial $\tilde{\Delta}(q^{-1})$ are chosen by the user. $\tilde{\Delta}(q^{-1}) = 1 - q^{-1}$ gives differential input penalty. In general, if $\tilde{\Delta}(q^{-1})$ is a high pass filter, high frequency components of the input will be penalized.

Simple calculations give the following input and output signals when (2.2) is used to control (2.1):

$$y(t) = \frac{MG}{P\alpha H} v(t) + \frac{RC}{\alpha} e(t) \qquad (2.5)$$

$$u(t) = -\frac{FG}{P\alpha H} v(t) - \frac{SC}{\alpha} e(t) \qquad (2.6)$$

where

$$\alpha \triangleq AR + q^{-k}BS \qquad (2.7)$$

$$M \triangleq q^{-d}DRP - q^{-k}BQ \qquad (2.8)$$

$$F \triangleq AQ + q^{-d}DSP. \qquad (2.9)$$

The characteristic polynomial of the closed loop system, including the disturbance model, is $P\alpha H$. The polynomial α is required to be stable. It is evident from (2.8) that the feedforward regulator $Q/P = q^{-d+k}DR/B$ would cancel the measurable disturbance. It is realizable (stable and causal) when $d \geq k$ and B is stable.

Remarks on the notation. The following different types of polynomials will be of use;

- $D \triangleq D(z) = d_0 + d_1 z + \cdots + d_{nd} z^{nd}$, where z is used for q^{-1}
- *conjugate polynomials:* $D_* \triangleq D(z^{-1}) = d_0 + d_1 z^{-1} + \cdots + d_{nd} z^{-nd}$
- *reciprocal polynomials:* $\bar{D} \triangleq z^{nd} D_* = d_0 z^{nd} + d_1 z^{nd-1} + \cdots + d_{nd}$.

The polynomial arguments (q^{-1}, z or z^{-1}) are often omitted to simplify the notation. The zeros of $\bar{D}$ are the zeros of D, reflected in the unit circle. If D is stable, $\bar{D}$ will be unstable. D_s and D_u denote the stable and unstable parts of the polynomial D.

3. *Optimal feedforward control*

Let us define a spectral factorization

$$r\beta\beta_* = BB_* + \rho A\tilde{\Delta}\tilde{\Delta}_* A_* \qquad (3.1)$$

where r is a positive scalar factor and β is a stable monic polynomial in z.

Let $B = cB_s B_u$ where c is a constant, B_s is stable and monic and B_u is unstable with $\bar{B}_u$ being stable and monic. When *minimum variance control,* $\rho = 0$, is used, the spectral factor will be given by

$$\beta = B_s \bar{B}_u. \qquad (3.2)$$

To assure that β is stable, we require that B has no zeros on the unit circle when minimum variance control problems are considered.

The following theorem presents a method for optimizing the feedforward part (Q, P) of the regulator (2.2). The feedback (R, S), i.e. the parameters $(r_1, \ldots, r_{nr}, s_0, \ldots, s_{ns})$ in (2.3), are held fixed. A stable causal optimal feedforward filter Q/P can be computed for any stabilizing feedback (R, S).

Theorem 1: *Optimal feedforward control.* Consider the system (2.1) controlled by a stabilizing regulator (2.2) with fixed R, S. Feedforward filter parameters attaining the global minimum value of the criterion (2.4) with respect to $(p_1, \ldots, p_{np}, Q_0, \ldots, Q_{nQ})$ are given by

$$P = G\beta \qquad (3.3)$$

where β is the stable spectral factor of (3.1). The regulator polynomial Q and a polynomial L (giving the minimal criterion value) are the solution of

$$(BR_* - \rho z^{-k} A\tilde{\Delta}\tilde{\Delta}_* S_*) z^{-d+k} D_* G_* = r\beta Q_* + \alpha_* H_* zL. \qquad (3.4)$$

If $\Lambda_e = 0$ (only measurable disturbances), the minimal criterion value $2J_{FF}$ is

$$2J_{FF} = \frac{\Lambda_v}{2\pi i} \oint \frac{LL_*}{r\beta\beta_*} \frac{dz}{z} + \frac{\rho\Lambda_v}{2\pi i} \oint \frac{GG_* DD_* \tilde{\Delta}\tilde{\Delta}_*}{r\beta\beta_* HH_*} \frac{dz}{z} \qquad (3.5)$$

■

Proof. See Appendix.

Remarks and interpretations

- Since β (stable) and $\alpha_* H_*$ (unstable) cannot have common factors, (3.4) will be solvable. The degrees of $Q_*(z^{-1})$ and $L(z)$ are chosen so that the maximal occurring powers in z^{-1} and z, respectively, are covered. The degree of L should e.g. be max $\{n\beta, nb - d + k\} - 1$.
- It is simple to generalize the design to systems with one input and several measurable disturbances $w_i(t) = (G_i/H_i)v_i(t)$. Use a regulator with feedforward links $(Q_i/P_i)w_i(t)$ calculated from

$$P_i = \beta G_i \qquad (3.6)$$

$$(BR_* - \rho z^{-k} A\tilde{\Delta}\tilde{\Delta}_* S_*) z^{-d_i+k} D_{i*} G_{i*} = r\beta Q_{i*} + \alpha_* H_{i*} z L_i. \qquad (3.7)$$

- In the minimum variance control case ($\rho = 0$), $P = \beta G = B_s \bar{B}_u G$ has a straightforward interpretation: if the system has minimum phase zeros, they are cancelled. Non-minimum phase zeros should of course remain uncancelled. The optimal feedforward filter has poles in their *inverse points with respect to the unit circle.* In addition, regulator poles should cancel the (stable) zeros of the disturbance model G/H. (Cancellation of stable zeros of B on the negative real axis may, however, lead to an oscillative input, and hidden inter-sample oscillations on the output. Use of a small input penalty solves this problem.)
- When perfect feedforward is impossible, the optimal regulator will depend on our disturbance model: G and H enter into (3.3) and (3.4). [When $\rho = 0$, $d \geq k$ and B is stable, (3.3) and (3.4) reduce to the perfect feedforward regulator $Q/P = q^{-d+k}DR/B$, $L = 0$. The disturbance spectrum then becomes irrelevant.]

With the procedure in Theorem 1, feedforward filters may be optimized for any given system controlled with a prespecified stabilizing feedback. We may ask how the choice of feedback (R, S) affects the achievable feedforward control performance.

Theorem 2. The minimal criterion value (3.5) for $e(t) = 0$ is independent of the choice of feedback (R, S) in (2.2) if $\alpha = AR + q^{-k}BS$ is stable.

Proof. Suppose that the system

$$Ay(t) = q^{-k}Bu(t) + q^{-d}Dw(t) \quad (e(t) = 0) \tag{3.8}$$

is controlled by

$$R_0u(t) = -\frac{Q_0}{P_0}w(t) - S_0y(t) \tag{3.9}$$

for some stable P_0 and stabilizing (R_0, S_0).

Then the same stationary output and input, and consequently the same criterion value, may be achieved by the regulator

$$Ru(t) = -\frac{Q}{P}w(t) - Sy(t) \tag{3.10}$$

where (R, S) stabilize the system and P, Q are calculated from

$$\frac{Q}{P} = \frac{Q_0\alpha - q^{-d}DP_0(R_0S - S_0R)}{P_0\alpha_0} \tag{3.11}$$

where $\alpha = AR + q^{-k}BS$ and $\alpha_0 = AR_0 + q^{-k}BS_0$ are stable.

We may verify this by using the control law (3.10) on the system (3.8), with Q/P chosen according to (3.11).

This gives the closed loop system

$$\alpha P_0\alpha_0 y(t) = (q^{-d}DR_0P_0 - q^{-k}BQ_0)\alpha w(t)$$
$$\alpha P_0\alpha_0 u(t) = -(AQ_0 + q^{-d}DS_0P_0)\alpha w(t).$$

The modes corresponding to α are hidden. They are neither visible from the input nor from the output. Thus, the transfer functions are the same as if the regulator (3.9) had been used for control. Compare with (2.5)–(2.9).

We conclude that there cannot exist a feedback R_0, S_0 which makes it possible to achieve a lower criterion value than with other feedbacks R, S. If (R_0, S_0, Q_0, P_0) is optimal, the regulator (3.10), (3.11) will achieve the same J. ∎

The feedback will affect the transient from an initial condition. In stationarity however, the feedback neither improves nor impairs the feedforward quality. This makes it possible to solve the feedback and feedforward design problems separately. The transformation (3.11) was suggested by E. Trulsson (1985, private communication). With other choices of regulator structures, the achievable feedforward performance depends on the choice of feedback. For example, consider control error feedback, where the feedforward signal enters through the reference input:

$$u(t) = \frac{S}{R}(y_r(t) - y(t)); \quad y_r(t) = -\frac{Q}{P}w(t).$$

Such regulators introduce additional zeros (the zeros of S) into the path from y_r to y. This may impair the feedforward performance, in particular if these zeros are unstable. The same is true for regulators where a feedforward compensation signal is added to the feedback control signal:

$$u(t) = -\frac{S}{R}y(t) + u_{FF}(t); \quad u_{FF}(t) = -\frac{Q}{P}w(t).$$

This regulator structure introduces additional zeros (the zeros of R) into the path from u_{FF} to y. With the structure we have chosen, (2.2), introduction of new zeros into the feedforward path by the feedback is avoided.

Assume that the system is stable, the unmeasurable disturbances are insignificant and a precise model of the system is available. Then, feedforward may be used without feedback.

Corollary 3: *Feedforward control of possibly non-minimum phase systems.* Under the constraint of stability and causality, the feedforward regulator

$$u(t) = -\frac{Q(q^{-1})}{P(q^{-1})}w(t)$$

attains the global minimum value of the criterion J for a stable system (2.1) if $P = \beta G$ and $Q_*(z^{-1})$, $L(z)$ are the minimal degree solution of

$$z^{-d+k}BD_*G_* = r\beta Q_* + A_*H_*zL. \tag{3.12}$$

∎

Proof. This follows immediately from Theorem 1 with $R = 1$, $S = 0$, and $\alpha = A$.

Corresponding feedforward regulators could be constructed with state space methods. Use of an algebraic Riccati equation requires about the same amount of computation as a spectral factorization, if the state vector dimension equals the polynomial order. In many cases, however, the polynomial approach leads to simpler calculations. This is especially evident for systems with significant time delays. Delays increase the dimension of the state vector, and the computational burden in solving Riccati equations. Compare this to (3.1) and (3.12). The spectral factorization is unaffected by the delays k and d. The order of the polynomial equation is affected only by the *difference* $k - d$.

4. *LQG-optimal combined feedback and feedforward regulators*

Normally, feedforward has to be used in combination with feedback, for several reasons: the feedforward control principle is not robust. Significant unmeasurable disturbances may be present. In addition, the system may be unstable.

A method is now presented for optimizing combined feedback–feedforward regulators with the structure (2.2).

Theorem 4: *Optimal combined feedback and feedforward.* For the system (2.1), the global minimum value of the criterion (2.4) with respect to the parameters (2.3) of a stabilizing and causal regulator (2.2) is attained, if

$$P = G \tag{4.1}$$

and the regulator polynomials R, S and Q are calculated as follows.

Let β be the stable spectral factor from (3.1):

$$r\beta\beta_* = BB_* + \rho A\tilde{\Delta}\tilde{\Delta}_*A_*.$$

Let $R_*(z^{-1})$, $S_*(z^{-1})$ and $X(z)$ be the minimum degree solution with respect to X of the coupled polynomial equations

$$\begin{cases} r\beta R_* - z^{-k+1}B_*X = \rho\tilde{\Delta}\tilde{\Delta}_*AC_* & (4.2a) \\ r\beta S_* + zA_*X = z^kBC_*. & (4.2b) \end{cases}$$

Let $Q_*(z^{-1})$ and $L(z)$ be the minimum degree solution of the polynomial equation

$$z^{-d+1}D_*G_*X = r\beta Q_* + C_*H_*zL. \tag{4.3}$$

Proof. see Appendix

Note that P is given by G above, while $P = \beta G$ has to be used when a feedback is absent or prespecified in an arbitrary way. Use of an optimal feedback allows us to use a lower order feedforward filter. (As a consequence, Theorem 1 and Corollary 3 cannot be derived as special cases of Theorem 4.) Equation (4.2) implies $\alpha = AR + q^{-k}BS = \beta C$.

Let us describe the optimization algorithm in words.

- First, the stable spectral factor $\beta(z)$ is calculated.
- Then, the feedback part R, S is optimized with respect to the unmeasurable disturbance $e(t)$. The poles are placed in $\alpha = \beta C$. This calculation is totally independent of the feedforward filter and the transfer functions in the measurable disturbance path. This means that the regulator structure has *two degrees of freedom*.
- The feedforward filter is then calculated to suppress the measurable disturbance $w(t)$ in an optimal way. The filter will depend on the feedback calculated in the preceding step [$X(z)$ enters into (4.3)]. However, as was stated in Theorem 2, the achievable feedforward performance is not affected by the choice of feedback with our regulator structure.

Placement of poles in C may lead to nonrobust feedbacks, with small gain and phase margins from some C-polynomials. (Poles in C correspond to the Kalman filter observer poles of a state space design.) This is a drawback of the LQG design. A suboptimal pole placement should be used in such cases. The sensitivity of the other poles, in β, can in general be decreased by increasing ρ, see also Nordström (1987).

The delay d affects the achievable control quality significantly. Complete cancellation of $w(t)$ is possible only when $d \geq k \geq 1$. It can be shown that when $d = 0$, use of feedforward does not improve the achievable control performance, compared to feedback from $y(t)$ only. When $d > 0$, there will always be an improvement. Thus, it is important to place the auxiliary $w(t)$-sensor so that the disturbance is captured as early as possible, i.e. d is large.

For a somewhat different system structure, combined optimization of feedback and feedforward has been treated by Grimble (1986) (for $d = 0$ only) and Hunt *et al.* (1987) (for $d \geq 0$). For the special case $C = 1$, $G = 1$, the regulator of Theorem 4 has been derived by Peterka (1984). In that work, two coupled polynomial equations define the feedforward polynomial Q. Only one equation, namely (4.3), is in fact needed.

The regulator (2.2) must be realized minimally, as a single dynamical system having two inputs and one output. For spectral factorization algorithms, see Kučera (1979). For $n\beta \leq 2$, simple explicit expressions exist for the spectral factor, cf. Peterka (1984). The coupled equation (4.2) can be found to give an over-determined set of simultaneous equations in the coefficients of R, S and X. This system will, however, have a unique minimum degree solution w.r.t X. (Some equations of the system will be linear combinations of the others.) A simple way to find the (exact) solution is to use the least-squares method for solving over-determined systems of equations.

For a further discussion of the regulator above, the interested reader is referred to Sternad (1987). There, servo problems are also discussed. Step disturbances and drifting disturbances are handled by using a differential model, as in Peterka (1984). Auxiliary signals $w(t)$ affected by the input $u(t)$ are handled by subtracting the input influence internally, inside the regulator. It is also shown that the feedforward regulator of Corollary 3 cancels deterministic disturbances (steps, ramps, sinusoids) asymptotically, without further modification. Several adaptive implementations have been tested and compared.

5. *Concluding discussion*

We have presented a polynomial LQG approach for designing feedforward regulators. Scalar discrete-time systems have been considered. Given a linear system and possibly a prespecified feedback, optimal feedforward filters for non-minimum phase systems can be calculated in a simple way. By penalizing a filtered input signal, it is possible to make frequency-dependent trade-offs between input energy and disturbance rejection. Optimal combined feedback and feedforward regulators can be designed by a simple step-by-step method.

The results have some limitations. Only linear single input systems have been considered. The use of an infinite horizon criterion neglects the effect of initial values. When β has zeros close to the unit circle, the regulator start-up from an initial condition will be slow. Time-varying regulators could provide an optimal transient, but they would be considerably more complicated than the ones discussed above.

A discussion of several other aspects, such as the control behaviour between the sampling instants and problems with noise-corrupted measurements, can be found in Sternad (1987). A very close correspondence has been found between optimal feedforward control and the input estimation (deconvolution) problem. The correspondence is explained in Sternad and Ahlén (1988).

The opportunities to use off-line designed feedforward regulators are limited by the availability of accurate process models. This motivates the development of adaptive regulators based on the algorithms presented above.

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Appendix

Proof of theorem 1

Since $v(t)$ and $e(t)$ are assumed to be mutually uncorrelated, the feedforward design is not affected by $e(t)$. Assume $e(t) = 0$. The expression (3.5) for the criterion value when the feedforward filter (3.3), (3.4) is used and $e(t) = 0$ has been derived from (2.5), (2.6) by straightforward, but somewhat tedious calculations in Sternad (1987). There, an alternative, more direct proof of optimality can be found. Here, we will verify that (3.5) represents the minimum, with a technique used for feedback optimization in Åström and Wittenmark (1984).

Let us write an arbitrary feedforward control action as

$$R(q^{-1})u(t) = -S(q^{-1})y(t) - \frac{Q(q^{-1})}{P(q^{-1})}w(t) + n(t) \quad \text{(A1)}$$

where R, S are prespecified and stabilizing, Q/P is calculated from Theorem 1, and $n(t)$ is an arbitrary additional feedforward signal, generated from a linear combination of measurements $w(t)$. It will be shown that it is optimal to choose $n(t) = 0$.

Using (2.1) with $e(t) = 0$, (A1), (2.7), (2.8) and (2.9), the criterion (2.4) can be expressed as

$$J = J_1 + 2J_2 + J_3 \quad \text{(A2)}$$

where

$$J_1 = \lim_{N\to\infty} \frac{1}{2N} \sum_{t=0}^{N} E\left(\frac{MG}{P\alpha H}v(t)\right)^2 + \rho E\left(\frac{\tilde{\Delta}FG}{P\alpha H}v(t)\right)^2$$

$$J_2 = \lim_{N\to\infty} \frac{1}{2N} \sum_{t=0}^{N} E\left(\frac{B}{\alpha}n(t-k)\frac{MG}{P\alpha H}v(t)\right) - \rho E\left(\frac{\tilde{\Delta}A}{\alpha}n(t)\frac{\tilde{\Delta}FG}{P\alpha H}v(t)\right)$$

$$J_3 = \lim_{N\to\infty} \frac{1}{2N} \sum_{t=0}^{N} E\left(\frac{B}{\alpha}n(t-k)\right)^2 + \rho E\left(\frac{\tilde{\Delta}A}{\alpha}n(t)\right)^2.$$

Because (Q, P) satisfy Theorem 1, the criterion value for $n(t)=0$, J_1, is finite and is given by (3.5). If $n(t)$ were nonstationary, the ensemble means in J_3 and J_2 would change with time, and the criterion could be undefined. Assume $n(t)$ to be stationary. It can then be expressed as a filtered feedforward signal

$$n(t)=\frac{T(q^{-1})}{N(q^{-1})}w(t)=\frac{T(q^{-1})}{N(q^{-1})}\frac{G(q^{-1})}{H(q^{-1})}v(t) \tag{A3}$$

where N is stable. Using Parseval's formula, (3.3), (2.8), (2.9) and (A3), the term J_2 can be expressed as

$$\begin{aligned}J_2 &= \frac{1}{2\pi i}\oint_{|z|=1}\frac{z^kBTGM_*}{\alpha NH\beta_*\alpha_*H_*}\frac{\mathrm{d}z}{z} - \frac{\rho}{2\pi i}\oint_{|z|=1}\frac{\tilde{\Delta}ATG}{\alpha NH}\frac{\tilde{\Delta}_*F_*}{\beta_*\alpha_*H_*}\frac{\mathrm{d}z}{z}\\ &= \frac{1}{2i}\oint\frac{TG[(z^kB(z^{-d}D_*R_*G_*\beta_*-z^{-k}B_*Q_*) - \rho\tilde{\Delta}\tilde{\Delta}_*A(A_*Q_*+z^{-d}D_*S_*G_*\beta_*)]}{\alpha NH\beta_*\alpha_*H_*}\frac{\mathrm{d}z}{z}\\ &= \frac{1}{2\pi i}\oint\frac{TG[(z^{k-d}BR_*-z^{-d}\rho A\tilde{\Delta}\tilde{\Delta}_*S_*)D_*G_*\beta_* - r\beta\beta_*Q_*]}{\alpha NH\beta_*\alpha_*H_*}\frac{\mathrm{d}z}{z}\end{aligned}$$

where the spectral factorization (3.1) was used in the last step. The use of the polynomial equation (3.4) now reduces J_2 to

$$J_2=\frac{1}{2\pi i}\oint\frac{TG\beta_*\alpha_*H_*zL}{\alpha NH\beta_*\alpha_*H_*z}\mathrm{d}z=\frac{1}{2\pi i}\oint\frac{T(z)L(z)G(z)}{\alpha(z)N(z)H(z)}\mathrm{d}z=0$$

since $\alpha(z)$, $N(z)$ and $H(z)$ are assumed to be stable, i.e. to have poles strictly outside the unit circle. Thus, the integral vanishes because the integrand has no poles inside the integration path.

Now, since $J_2=0$, and J_3 is quadratic in $n(t)$, it is obvious that (A2) is minimized by setting $n(t)=0$, i.e. by using the feedforward filter defined by Theorem 1.

■

Proof of Theorem 4

As a consequence of Theorem 2, a feedback (R, S) can first be designed optimal with respect to $e(t)$. Then, a feedforward filter optimizing the influence from $w(t)$ can be designed from Theorem 1. As was proved by Kučera (1979), an optimal feedback is designed by solving (4.2). For an alternative proof, see Peterka (1984). By multiplying (4.2a) by A_* and (4.2b) by $z^{-k}B_*$ and adding them, (4.2) is seen to imply pole placement in $\alpha=\beta C$. Now, multiply $\alpha_*=A_*R_*+z^{-k}B_*S_*=\beta_*C_*$ by $\rho A\tilde{\Delta}\tilde{\Delta}_*$. This gives

$$-\rho A\tilde{\Delta}\tilde{\Delta}_*z^{-k}B_*S_*=\rho A\tilde{\Delta}\tilde{\Delta}_*A_*R_*-\rho A\tilde{\Delta}\tilde{\Delta}_*\beta_*C_*.$$

Addition of BB_*R_* and multiplication by z^{k-1} gives

$$z^{k-1}(BB_*R_*-\rho A\tilde{\Delta}\tilde{\Delta}_*z^{-k}B_*S_*) = z^{k-1}(r\beta\beta_*R_*-\rho A\tilde{\Delta}\tilde{\Delta}_*\beta_*C_*).$$

By dividing by β_*B_* and comparing with (4.2a) it is found that

$$\frac{z^{-1}(z^kBR_*-\rho A\tilde{\Delta}\tilde{\Delta}_*S_*)}{\beta_*}=\frac{z^{k-1}(r\beta R_*-\rho\tilde{\Delta}\tilde{\Delta}_*AC_*)}{B_*}=X(z). \tag{A4}$$

The optimal feedforward polynomial Q satisfies (3.4). The use of $\alpha_*=\beta_*C_*$ and (A4) in (3.4) gives

$$(z\beta_*X)z^{-d}D_*G_*=r\beta Q_*+\beta_*C_*H_*zL. \tag{A5}$$

Since β_* is a factor of the other two terms, it must be a factor of Q_*. (It cannot have common factors with β, since $z^{n\beta}\beta_*=\bar{\beta}$ will be strictly unstable, while β is stable.) Let $Q_*=Q'_*\beta_*$.

Equation (A5) then reduces to (4.3)

$$z^{-d+1}D_*G_*X=r\beta Q'_*+C_*H_*zL.$$

If the stable common factor β in $Q/P=Q'\beta/G\beta$ is cancelled and Q' is named Q, we have the regulator (4.1), (4.3).

■