Realizable MIMO Decision Feedback Equalizers: Structure and Design

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Abstract— **We present and discuss the structure and design of optimum multivariable decision feedback equalizers (DFEs). The equalizers are derived under the constraint of realizability, that is, causal and stable filters and finite decision delay. The design is based on a known dispersive discrete-time multivariable channel model, with infinite impulse response. The additive noise is described by a multivariate ARMA model. Equations for obtaining minimum mean square error (MMSE) and zero-forcing DFEs are derived, under the assumption of correct past decisions. Design of a realizable MMSE DFE requires the solution of a linear system of equations in the model parameters. No spectral factorization is required.**

We derive novel necessary and sufficient conditions for the existence of zero-forcing DFEs, and point out the significance of these conditions for the design of multiuser detectors.

An optimal MMSE DFE will contain IIR filters in general. Simulations indicate that the performance improvement obtained with an IIR DFE is reduced more than for a (suboptimal) FIR DFE when error propagation is taken into account, since the use of IIR feedback filters tends to worsen the error propagation.

I. Introduction

DURING the last three decades, *decision feedback equalizers (DFEs)* have been used in digital communications to suppress *intersymbol interference (ISI)*. The DFE provides a good compromise between performance and complexity: It gives much better performance than a linear equalizer, and it has a much lower complexity than the optimum detector, the maximum likelihood sequence estimator (MLSE).

A (scalar) DFE is a symbol-by-symbol detector which consists of two linear filters and a decisions non-linearity. The ISI-corrupted measurements are processed by the *feedforward filter*. Already detected symbols are fed back via a *feedback filter* and subtracted from the feedforward filter output to provide a soft symbol estimate. A hard decision is then made to decide what symbol was transmitted. This decision is fed into the feedback filter to remove its effect on future symbol estimates. The filter coefficients are mostly adjusted according to either the zero-forcing (ZF) or the minimum mean square error (MMSE) criterion. With a zero-forcing equalizer, all intersymbol interference is to be removed, whereas an MMSE equalizer minimizes the mean square difference between the transmitted symbols and their soft estimates.

In the literature [1], two types of MMSE DFEs have been proposed:

1. Optimal model based design, which results in a continuous-time non-causal feedforward filter. This structure is optimized based on a known noise spectrum and transfer function of the communication channel.
2. Fixed structure design, where the DFE often has FIR filters of predetermined degrees in both the feedforward and feedback paths. The coefficients of these filters are determined by solving the Wiener-Hopf equations.

The performance of the non-causal DFE is always better than that of a realizable (causal and stable) DFE, with finite decision delay. The fixed structure DFE on the other hand may have a suboptimal structure or incorrect filter degrees, resulting in a sub-optimum performance.

The structural problem was resolved in [2], where an *optimum realizable* DFE was derived based on discrete time IIR models of the channel and the noise. The resulting DFE had optimal structure with optimal filter degrees and design equations on closed form were also presented.

During the last few years, channels with several inputs and/or outputs have gained increased interest. Such channels occur in many areas, for example in cellular communication systems where antenna arrays are used to improve the detection. Oversampled received signals can also be regarded as generated by time-invariant channels with several outputs. With a detector based on a model with multiple inputs and/or outputs, it is possible to suppress not only intersymbol interference, but also co-channel interference, the interference from other sources.

A *multiple input-multiple output (MIMO)* DFE is a DFE where both the feedforward and the feedback filter have multiple inputs and multiple outputs. The DFE is an attractive compromise between complexity and performance also in the MIMO case. As in the scalar case, studies of MIMO DFEs have been based on one of two principles: Either a model-based design of a DFE with a non-causal feedforward filter [3] or a model- or data-based optimization of DFEs whose structure is fixed prior to the design [4], [5], [6].

In this paper, we provide tools for model-based DFE designs and for understanding of their structural properties. In Section IV, we present a generalized DFE with several inputs and outputs, which minimizes the mean square error under the constraint of realizability. The design is based on a multivariable discrete-time IIR channel model and a multivariate ARMA noise description. An MMSE optimal DFE based on these assumptions will in general have multivariable IIR filters in both the feedforward and feedback paths. These filters will have optimal degrees with parameters obtained from closed-form design equations. In the limit, when the decision delay tends to infinity, we also obtain the non-realizable MMSE DFE. We derive novel necessary and sufficient conditions for the existence of zero-forcing DFEs, and point out the significance of these conditions for the design of multiuser detectors. For example, they indicate well-behavedness, and near-far resistance of MMSE equalizers designed for low noise levels.

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Moreover, the conditions for the existence of ZF solutions are much milder for DFEs than for linear equalizers.

An MMSE optimal GDFE will contain IIR filters in general. Simulations in Section V indicate that the performance obtained with an IIR DFE is reduced more than for a (suboptimal) FIR DFE when error propagation is taken into account, since the use of IIR feedback filters tends to worsen the error propagation.

A. *Notations*

Throughout the paper, channels and filters are assumed to be linear and time-invariant. A scalar discrete-time channel or filter will be represented by a rational function $\mathcal{H}(q^{-1})$ in the unit delay operator q^{-1}, or as a ratio of polynomials in q^{-1}:

$$s(k) = \mathcal{H}(q^{-1})u(k) = \frac{B(q^{-1})}{A(q^{-1})}u(k) \ .$$

When appropriate, the complex variable z will be substituted for the forward shift operator q. For convenience, polynomial arguments will often be omitted when there is no risk of misunderstanding.

Multivariable channels, or MIMO channels, will be described by *rational matrices*, that is, matrices whose elements are rational functions. We can expand any stable and causal rational matrix $\boldsymbol{\mathcal{T}}(q^{-1})$ in the series

$$\boldsymbol{\mathcal{T}}(q^{-1}) = \sum_{n=0}^{\infty} \mathbf{T}_n q^{-n} \tag{1a}$$

where $\mathbf{T}_n$ are the Markov parameters of $\boldsymbol{\mathcal{T}}(q^{-1})$. For any such rational matrix, we define

$$\boldsymbol{\mathcal{T}}_*(q) \triangleq \sum_{n=0}^{\infty} \mathbf{T}_n^H q^n \tag{1b}$$

where $(\cdot)^H$ denotes complex conjugate transpose. We will also consider rational matrices which are non-causal. Such a matrix can be expanded in the series

$$\boldsymbol{\mathcal{T}}(q, q^{-1}) = \sum_{n=-\infty}^{\infty} \mathbf{T}_n q^{-n} \ . \tag{2a}$$

In this case, the definition (1b) can be generalized to

$$\boldsymbol{\mathcal{T}}_*(q, q^{-1}) \triangleq \sum_{n=-\infty}^{\infty} \mathbf{T}_n^H q^n \ . \tag{2b}$$

The degree of a *polynomial matrix* $\boldsymbol{P}(q^{-1})$

$$\boldsymbol{P}(q^{-1}) \triangleq \mathbf{P}_0 + \mathbf{P}_1 q^{-1} + \cdots + \mathbf{P}_{\delta P} q^{-\delta P} \tag{3a}$$

equals the highest degree of any of its elements, and is denoted δP. For any polynomial matrix (3a), we also define

$$\boldsymbol{P}_*(q) \triangleq \mathbf{P}_0^H + \mathbf{P}_1^H q + \cdots + \mathbf{P}_{\delta P}^H q^{\delta P} \tag{3b}$$

$$\bar{\boldsymbol{P}}(q^{-1}) \triangleq q^{-\delta P}\boldsymbol{P}_*(q) \tag{3c}$$

$$= \mathbf{P}_{\delta P}^H + \mathbf{P}_{\delta P-1}^H q^{-1} + \cdots + \mathbf{P}_0^H q^{-\delta P} \ .$$

A square rational matrix $\boldsymbol{P}^{-1}(q^{-1})$ is called stable if $\det \boldsymbol{P}(z^{-1})$ has all its zeros inside $|z| = 1$.

II. Multivariable channel models

We shall consider communication systems with receiver front-ends as depicted in Figure 1. The in-phase and quadrature components of the received passband signal $r(t)$ are down-converted to the baseband. The baseband signal is passed through a fixed anti-aliasing filter and sampled. Depending on the application, the sampling rate either equals, or is a multiple of, the symbol rate. When designing the detectors, the transmitter filter, the multipath channel and the receiver filter are lumped together, and the resulting discrete-time channel from the transmitted symbols to the received sampled signal is used as a basis for detector design.

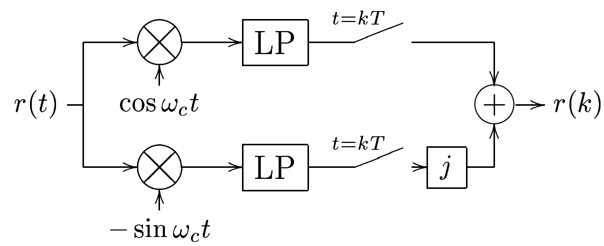

Fig. 1. The front-end of a receiver in the considered communication system.

All signals are represented by their complex envelopes and the coefficients in the discrete-time channel models will in general be complex-valued.

Remark 1. As is apparent from Figure 1, the receiver front-end incorporates no filter matched to the received signal, as is common in optimal, model-based detector design [7]. This means that there is no guarantee that the sampled signal constitutes a sufficient statistic of the original continuous-time signal $r(t)$. Also, the noise suppression of the anti-aliasing filter is worse than that of an ideal matched filter. Still, we will not use a matched filter as an *a priori* component in our detector design. The reason for this is threefold:

1. As explained in Section IV.C, matched filtering may not be optimal for a detector with finite smoothing lag.
2. For channels having infinite impulse responses (IIR channels), the matched filter would not be realizable.
3. In a practical communication system, a fixed analog filter must be used.

□

With the receiver front-end depicted in Figure 1, the resulting channel is scalar, baseband and discrete-time. We now collect the outputs from n_y such channels in a vector

$$y(k) \triangleq \begin{pmatrix} y_1(k) & y_2(k) & \ldots & y_{n_y}(k) \end{pmatrix}^T \ . \tag{4}$$

Assume that the channel output is affected by n_d input signals. These n_d signals are stacked in a vector

$$d(k) \triangleq \begin{pmatrix} d_1(k) & d_2(k) & \ldots & d_{n_d}(k) \end{pmatrix}^T \ , \tag{5}$$

where the signal $d_j(k)$ is transmitted from user j. We now assume that a multiple input-multiple output (MIMO) model of the channel in the form

$$y(k) = \boldsymbol{\mathcal{H}}(q^{-1})d(k) + w(k) \tag{6}$$

has been made available. Here, the $n_y \times n_d$ channel matrix $\boldsymbol{\mathcal{H}}(q^{-1})$ may be a rational (IIR) transfer function matrix, which is assumed to be causal and stable. It is also assumed to be time-invariant over the time interval of interest. The noise vector

$$w(k) \stackrel{\triangle}{=} \begin{pmatrix} w_1(k) & w_2(k) & \dots & w_{n_y}(k) \end{pmatrix}^T \tag{7}$$

represents thermal noise and also the effect on $y(k)$ of all co-channel interferers which are not modeled explicitly via the input-output model $\boldsymbol{\mathcal{H}}(q^{-1})d(k)$. It is assumed to be stationary and zero mean.

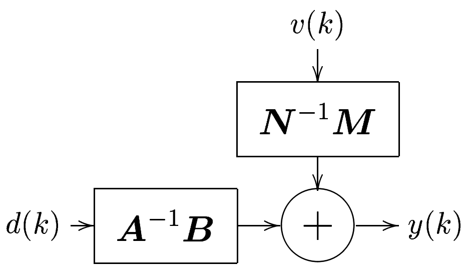

Fig. 2. The multivariable system model. Both the IIR channel and the ARMA noise model are parametrized as left matrix fraction descriptions (MFDs).

We now assume the channel matrix $\boldsymbol{\mathcal{H}}(q^{-1})$, and the vector ARMA model representing the noise $w(k)$ to be described by left matrix fraction descriptions (MFDs) [8]:

$$y(k) = \boldsymbol{A}^{-1}(q^{-1})\boldsymbol{B}(q^{-1})d(k) + \boldsymbol{N}^{-1}(q^{-1})\boldsymbol{M}(q^{-1})v(k) \; . \tag{8}$$

The polynomial matrix $\boldsymbol{B}(q^{-1})$ has n_y rows and n_d columns, whereas $\boldsymbol{A}(q^{-1})$, $\boldsymbol{M}(q^{-1})$ and $\boldsymbol{N}(q^{-1})$ are square polynomial matrices of dimension n_y. These three matrices are assumed to be *stably invertible*; the roots of

$$\det \boldsymbol{A}(z^{-1}) = 0 \; , \; \det \boldsymbol{M}(z^{-1}) = 0 \; , \; \det \boldsymbol{N}(z^{-1}) = 0 \tag{9}$$

are all located inside $|z| = 1$. For FIR channels, $\boldsymbol{A}(q^{-1}) = \mathbf{I}$, while $\boldsymbol{N}(q^{-1}) = \mathbf{I}$ for moving average noise models. For autoregressive noise models, $\boldsymbol{M}(q^{-1})$ is a constant matrix $\mathbf{M}_0$.

In general, we assume the leading matrix coefficient $\mathbf{M}_0$ of $\boldsymbol{M}(q^{-1})$ to be non-singular. The denominator matrices $\boldsymbol{A}(q^{-1})$ and $\boldsymbol{N}(q^{-1})$ are assumed to be monic (the leading matrix coefficient is equal to the identity matrix). To simplify the presentation, $\boldsymbol{A}(q^{-1})$ and $\boldsymbol{N}(q^{-1})$ are also assumed to be *diagonal*.[1] The polynomial elements in the matrices may have complex coefficients, and are assumed to be known *a priori* or correctly estimated.

Each element of the vector $d(k)$ is taken from a finite set of values, the symbol alphabet,

$$d_i(k) \in \mathcal{A}_i \; .$$

For instance, when binary phase shift keying (BPSK) is employed, $\mathcal{A}_i = \{+1, -1\}$. We assume each $d_i(k)$ to be a stochastic variable with zero mean, which is uncorrelated with the disturbance vector $v(k)$. Finally, $d_i(k)$ is assumed white[2] with covariance matrix

$$Ed(k)d^H(k) = \lambda_d \mathbf{I} \; . \tag{10}$$

The noise vector $v(k)$ is a white stochastic process with zero mean and covariance matrix

$$E[v(k)v^H(k)] = \lambda_v \mathbf{I} \; . \tag{11}$$

For future reference, we also define

$$\rho \stackrel{\triangle}{=} \frac{\lambda_v}{\lambda_d} \; . \tag{12}$$

Models of the type (6) appear when using, for example,

1. *Diversity receivers or antenna arrays* in digital mobile radio systems see, for example, [9], [10], [11] and [12], where the signal at receiver i is denoted $y_i(k)$ and the channel $\boldsymbol{\mathcal{H}}(q^{-1})$ is a column matrix of FIR channels, describing the multipath propagation from one user to each receiver. The aim is here to detect the message from the user and to reject the interference. A MIMO channel model (6), with $n_d > 1$, can furthermore be used for designing multiuser detectors, which simultaneously detect several symbol streams. This approach was first studied by Winters in [13] and more recently for TDMA/FDMA systems by Tidestav *et al.* in [6]. Such a MIMO model may also be used to describe a situation, where one user terminal is equipped with several transmitter antennas. Over each antenna, a different symbol stream can be transmitted. This will increase the peak rate of that user.
2. *Multiuser detection in DS-CDMA systems.* Here many users, distinguished by a user-specific spreading code, share the same channel. At the receiver, signals from the different users are separated, “despread”, by means of cross-correlation with the appropriate spreading code. Due to multipath propagation and asynchronism, the despread sequences will be mutually correlated, so they have to be further processed. Assume that (4) represents the symbol-sampled output from n_y correlators, and that short spreading codes are used. We then obtain a time-invariant symbol-sampled input-output model (6) for a system with $n_d = n_y$ simultaneous users [14], [5]. The channel $\boldsymbol{\mathcal{H}}$ will constitute a matrix of polynomials (FIR filters). This model can be used for designing decorrelating (zero-forcing) or MMSE-optimal multiuser detectors.
3. *Fractionally spaced sampling*, can be used to reduce the sensitivity of the receiver to synchronization errors, or simply to improve detector performance. When a scalar received baseband signal is sampled n_y times per symbol period, it will have cyclo-stationary statistics. We may stack n_y received samples at a time into a symbol-sampled vector $y(k)$, which will then have stationary

[1] This will result in less complex design equations, but might lead to unnecessarily high polynomial degrees in the matrix elements, since, neither $\boldsymbol{A}^{-1}\boldsymbol{B}$ nor $\boldsymbol{N}^{-1}\boldsymbol{M}$ constitute irreducible MFDs.

[2] In a communication system employing interleaving, this assumption is in general valid.

statistics. A time-invariant channel model (6) can then be obtained, where $\boldsymbol{\mathcal{H}}$ is a column vector of FIR filters (for wireless channels) or IIR filters (for wireline channels). When several signals are transmitted over a common channel, an oversampled version of the received signal can be used to detect all of them. This can be implemented in CDMA systems, as described in [15], [16], and in xDSL systems, as described in [17]. The resulting model will have multiple inputs, as well as multiple outputs.

III. Problem statement

Our goal is to reconstruct the sequence of symbol vectors $d(k)$ from the measurements $y(k)$. For this purpose, we introduce the general IIR decision feedback equalizer (GDFE)

$$\begin{aligned}\hat{d}(k-\ell|k) &= \boldsymbol{\mathcal{R}}(q^{-1})y(k) - \boldsymbol{\mathcal{F}}(q^{-1})\tilde{d}(k-\ell-1) \\ \tilde{d}(k-\ell) &= \mathrm{f}(\hat{d}(k-\ell|k)) \ .\end{aligned} \qquad (13)$$

In order to avoid introducing structural constraints, the feedforward filter $\boldsymbol{\mathcal{R}}(q^{-1})$ and the feedback filter $\boldsymbol{\mathcal{F}}(q^{-1})$ are assumed to be rational matrices. They are however required to be stable and causal. The design variable ℓ is known as the decision delay or the *smoothing lag* and represents the number of future measurements used to estimate the current symbol. The function $\mathrm{f}(\cdot)$ constitutes the decision non-linearity: For each element $\hat{d}_i(k-\ell|k)$ of the vector $\hat{d}(k-\ell|k)$, the decision device selects

$$\tilde{d}_i(k-\ell) = \arg\min_{d_i \in \mathcal{A}_i} |\hat{d}_i(k-\ell|k) - d_i|^2 \ .$$

The vector $\tilde{d}(k-\ell)$ thus constitutes the decision made on the estimate $\hat{d}(k-\ell|k)$. The GDFE is depicted in Fig. 3.

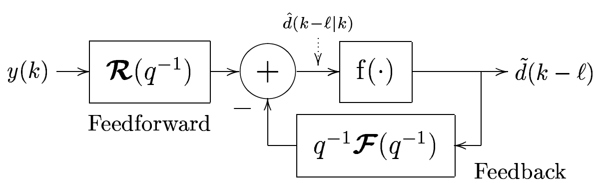

Fig. 3. The general IIR decision feedback equalizer (GDFE).

The GDFE (13) is assumed to be *realizable*. This constraint implies that

- the smoothing lag ℓ must be finite, and
- the filters must be causal and stable.

In Section IV.C, we relax the constraint of causality.

Given the received sequence $y(k)$ and the model (8), we seek the stable and causal linear time-invariant rational MIMO filters $\{\boldsymbol{\mathcal{R}}(q^{-1}), \boldsymbol{\mathcal{F}}(q^{-1})\}$ which minimize the estimation error covariance matrix[3]

$$\mathbf{P} \triangleq E\varepsilon(k-\ell)\varepsilon^H(k-\ell) \qquad (14)$$

[3]The covariance matrix is minimized in the sense that any other admissible choice of $\{\boldsymbol{\mathcal{R}}(q^{-1}), \boldsymbol{\mathcal{F}}(q^{-1})\}$ will result in an estimation error covariance matrix $\tilde{\mathbf{P}}$ such that $\tilde{\mathbf{P}} - \mathbf{P}$ is positive definite.

where the estimation error $\varepsilon(k-\ell)$ is defined as

$$\varepsilon(k-\ell) \triangleq d(k-\ell) - \hat{d}(k-\ell|k) \ . \qquad (15)$$

The GDFE which minimizes (14) will be called the *MMSE GDFE*.

We are also interested in finding the conditions under which a *zero-forcing* solution to the equalization problem exists. A scalar zero-forcing equalizer removes all intersymbol interference from the symbol estimate. A natural extension to the multivariable case [11] is to require that both the intersymbol interference and all co-channel interference from users explicitly included in the channel model should be removed. A multivariable zero forcing (ZF) equalizer can then be defined accordingly:

Definition 1: Consider the channel model (8) and a multivariable equalizer which forms the estimate $\hat{d}(k-\ell|k)$ of a transmitted symbol vector $d(k-\ell)$. If

$$\hat{d}(k-\ell|k) = d(k-\ell) - \varepsilon(k-\ell) \qquad (16)$$

where $\varepsilon(k-\ell)$ is uncorrelated with all transmitted symbol vectors $d(m)\ \forall\, m$, then the equalizer is said to be *zero-forcing*.

Because of the presence of the non-linear decision device in (13), closed form expressions for the parameters of the minimum mean square error or the zero-forcing GDFE cannot be found. To enable a useful tuning of the GDFE coefficients, we shall adopt the common assumption that all past decisions affecting the current estimate are correct, or equivalently

$$\tilde{d}(k-n) = d(k-n) \quad n = \ell+1, \ell+2, \ldots \ .$$

This assumption implies that we do not use the actual decisions in our design, but only in the implementation. When this assumption is violated, the estimates that are delivered by the equalizer will degrade, possibly causing additional decision errors. This phenomenon is known as *error propagation*. Considerable effort has been spent on deriving performance bounds for DFEs when error propagation are taken into account [18], [19], [20]. However, decision feedback equalizers designed based on correct past decisions generally work well, and more elaborate schemes that, for example, model the effect of decision errors as an extra additive noise term, mostly give only small improvements [21].

When all past decisions are assumed correct, minimizing (14) becomes a linear filtering problem and the DFE can be found by solving a system of linear equations.

IV. Optimum general decision feedback equalizers

Based on the known multivariable channel (8) we will now describe how to adjust the coefficients of the multivariable GDFE (13), so that the estimation error covariance matrix (14) is minimized. We also derive the conditions under which the GDFE can be tuned so that the zero-forcing condition (16) is satisfied.

A. *The optimum MMSE GDFE*

Introduce the following polynomial matrices:

$$\boldsymbol{\Gamma}(q^{-1}) \stackrel{\triangle}{=} \boldsymbol{A}(q^{-1})\boldsymbol{M}(q^{-1}) \quad (17a)$$
$$\boldsymbol{\tau}(q^{-1}) \stackrel{\triangle}{=} \boldsymbol{N}(q^{-1})\boldsymbol{B}(q^{-1}) \;. \quad (17b)$$

Also, define the polynomial matrices $\tilde{\boldsymbol{\Gamma}}(q^{-1})$ and $\tilde{\boldsymbol{\tau}}(q^{-1})$ by the matrix identity

$$\tilde{\boldsymbol{\tau}}(q^{-1})\tilde{\boldsymbol{\Gamma}}^{-1}(q^{-1}) = \boldsymbol{\Gamma}^{-1}(q^{-1})\boldsymbol{\tau}(q^{-1}) \;. \quad (18)$$

Without restriction, we assume $\tilde{\boldsymbol{\tau}}(q^{-1})\tilde{\boldsymbol{\Gamma}}^{-1}(q^{-1})$ to constitute an irreducible right MFD.[4]

We are now ready to formulate our main result.

Theorem 1: Consider a multivariable channel described by (8) and assume that the transmitted data are characterized by (10). Furthermore, let the second order moments of the noise be described by (11), (12) with a known $\rho > 0$. Assuming correct past decisions, the general multivariable DFE (13) minimizes the estimation error covariance matrix (14) if and only if

$$\boldsymbol{\mathcal{R}}(q^{-1}) = \boldsymbol{S}(q^{-1})\boldsymbol{M}^{-1}(q^{-1})\boldsymbol{N}(q^{-1}) \quad (19a)$$
$$\boldsymbol{\mathcal{F}}(q^{-1}) = \boldsymbol{Q}(q^{-1})\tilde{\boldsymbol{\Gamma}}^{-1}(q^{-1}) \;. \quad (19b)$$

Above, $\boldsymbol{S}$ and $\boldsymbol{Q}$, together with the polynomial matrices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ can be calculated as the unique solution to the two coupled polynomial matrix equations

$$\tilde{\boldsymbol{\Gamma}} - q^{\ell}\boldsymbol{S}\tilde{\boldsymbol{\tau}} + q^{-1}\boldsymbol{Q} = \boldsymbol{L}_{1*}\tilde{\boldsymbol{\Gamma}} \quad (20a)$$
$$q^{-\ell}\boldsymbol{L}_{1*}\boldsymbol{\tau}_{*} - \rho\boldsymbol{S}\boldsymbol{\Gamma}_{*} = q\boldsymbol{L}_{2*} \quad (20b)$$

where the degrees of the unknown polynomial matrices satisfy

$$\delta S = \ell \quad (21a)$$
$$\delta Q = \max(\delta\tilde{\Gamma}, \delta\tilde{\tau}) - 1 \quad (21b)$$
$$\delta L_1 = \ell \quad (21c)$$
$$\delta L_2 = \max(\delta\tau, \delta\Gamma) - 1 \;. \quad (21d)$$

The minimum covariance matrix of the estimation error of the general multivariable DFE is given by

$$\mathbf{P} = \lambda_d \sum_{n=0}^{\ell} \mathbf{L}_{1n}^H \mathbf{L}_{1n} + \lambda_v \sum_{n=0}^{\ell} \mathbf{S}_n \mathbf{S}_n^H \;. \quad (22)$$

Proof: See Appendix A. ■

Remark 1. The degrees listed in (21a)–(21d) are sufficiently high. When τ_0 (and hence $\tilde{\tau}_0$) has full rank, these degree conditions are also necessary. The solution (19a), (19b) is unique, apart from stable common factors in the filter elements, which affect the initial transient but do not affect the steady-state error covariance.

Remark 2. The two coupled *Diophantine equations* (20a) and (20b) are *unilateral*, since all the unknown polynomial matrices appear on the same (in this case left) side of their respective coefficient polynomial matrices. Solving these unilateral Diophantine equations corresponds to solving a block-Toeplitz system of linear equations, as demonstrated in Appendix A, equations (50)–(53).

Remark 3. The first term in (22) is caused by residual intersymbol and co-channel interference from the first ℓ taps in the equalized channel. The deviation of the reference tap from the identity matrix also contributes to the term. The last term in (22) is caused by the noise.

Remark 4. The smoothing lag ℓ is a design variable and should be chosen as a trade-off between complexity and performance. In general, the smoothing lag should be chosen so that “enough” signal power can be collected by the feedforward filter before a decision is made. □

The presented MMSE solution provides an *optimal DFE structure*. It is evident that under the conditions in Theorem 1, the conventional DFE structure, where both the feedforward and the feedback filters have finite impulse responses, is optimal *only* when $\boldsymbol{A}(q^{-1}) = \boldsymbol{M}(q^{-1}) = \mathbf{I}$. In other words, the channel must be described by a finite impulse response model, whereas the additive noise must be an autoregressive (or white) process.[5]

In addition to providing an optimal DFE structure and optimal filter degrees, Theorem 1 gives guidelines on how to choose the filter degrees in a conventional structure when the use of the optimal structure is deemed inappropriate. For instance, when the noise is described by a moving average model, Theorem 1 states that both the feedforward and feedback filters should have infinite impulse responses. In that case, the transversal feedforward filter in the conventional DFE structure should have a long impulse response, of length $> \ell$, particularly if the zeros of $\boldsymbol{M}(z^{-1})$ are located close to the unit circle. If, on the other hand, the noise is described by an autoregressive process, both the feedforward and the feedback filter of an optimum DFE will be FIR filters.

B. *The ZF GDFE*

The primary focus in the literature regarding equalizer design has been on MMSE equalizers. When noise is present, an MMSE equalizer will in general provide superior performance, as compared to a corresponding ZF equalizer. Therefore, in an actual implementation, an MMSE equalizer is in general preferable. However, zero-forcing equalizers can provide information about the performance of their MMSE counterparts. If a ZF solution exists, then the MMSE solution will approximate a ZF solution for small noise levels. If, on the other hand, a ZF solution does not exist, then the MSE of a MMSE design will not vanish, and the MMSE DFE may in fact provide an inadequate performance. These properties are indicative of the *near-far resistance* [22] of a MMSE multiuser detector [6]. We will therefore now study the conditions for existence of realizable ZF solutions.

[4] Since polynomial and rational matrices in general do not commute, this factorization is required to obtain a minimal right MFD from a left MFD. See, for example, [8].

[5] When $\rho = 0$, the MMSE GDFE may reduce to the ZF DFE discussed in Subsection IV.B. In this case, the conventional DFE structure is optimum also for IIR channels.

Theorem 2: Consider the multivariable channel model (8) with known MFDs and the general DFE (13). There exists a zero-forcing multivariable DFE, in the sense of Definition 1, if and only if there exist stable and causal rational matrices $\boldsymbol{\mathcal{R}}(q^{-1})$ and $\boldsymbol{\mathcal{F}}(q^{-1})$ such that

$$q^{-\ell}\mathbf{I} = \boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} - q^{-\ell-1}\boldsymbol{\mathcal{F}} \ . \tag{23}$$

Proof: See Appendix B. ■

Equation (23) may have several solutions. Hence, for a given channel, there may be many multivariable zero-forcing decision feedback equalizers. However, in some cases no solution to (23) will exist. The precise condition for this is stated in Lemma 1.

Lemma 1: There exists a solution to (23) if and only if the following two conditions are fulfilled:

- $\boldsymbol{B}(z^{-1})$ has normal rank n_d.
- The order of every zero of $\boldsymbol{B}(z^{-1})$ at $z=\infty$ is less than or equal to ℓ.

Proof: See Appendix C. ■

Two important situations when the conditions in Lemma 1 are not fulfilled are the following:

1. The number of channel outputs n_y is smaller than the number of channel inputs n_d. In this case, the normal rank of $\boldsymbol{B}(z^{-1})$ will be less than n_d.
2. The minimal delay in any channel of at least one of the users is larger than ℓ. This would imply that $\boldsymbol{B}(z^{-1})$ has at least one zero at $z=\infty$ of order at least $\ell+1$.

Remark 5. Additional insights into the problem of finding zero-forcing DFEs can be obtained by rewriting (23) as

$$q^{-\ell}\mathbf{I} = (\mathbf{I}+q^{-1}\boldsymbol{\mathcal{F}})^{-1}\boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} \triangleq \boldsymbol{\mathcal{H}}^{L}\boldsymbol{\mathcal{H}} \ . \tag{24}$$

with $\boldsymbol{\mathcal{H}}(q^{-1}) = \boldsymbol{A}^{-1}\boldsymbol{B}$ and $\boldsymbol{\mathcal{H}}^{L}(q^{-1}) = (\mathbf{I}+q^{-1}\boldsymbol{\mathcal{F}})^{-1}\boldsymbol{\mathcal{R}}$. Since both $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{R}}$ are required to be causal, $\boldsymbol{\mathcal{H}}^{L}(q^{-1})$ must be causal. However, $\boldsymbol{\mathcal{H}}^{L}(q^{-1})$ is not required to be stable: Even if $\boldsymbol{\mathcal{F}}$ is stable $(\mathbf{I}+q^{-1}\boldsymbol{\mathcal{F}})^{-1}$ may very well be unstable. Thus, the class of admissible $\boldsymbol{\mathcal{H}}^{L}$s is much larger for DFEs than for linear equalizers ($\boldsymbol{\mathcal{F}}=0$): Unstable zeros of $\boldsymbol{B}$ cannot be canceled by $\boldsymbol{\mathcal{R}}$, since $\boldsymbol{\mathcal{R}}$ is required to be stable.

Another reformulation of (23) also provides some additional insight. We realize that we can write

$$q^{-\ell}\mathbf{I} = \begin{pmatrix}\boldsymbol{\mathcal{R}} & \boldsymbol{\mathcal{F}}\end{pmatrix}\begin{pmatrix}\boldsymbol{A}^{-1}\boldsymbol{B} \\ -q^{-\ell-1}\mathbf{I}\end{pmatrix} \ . \tag{25}$$

We thus see that there exists a ZF GDFE whenever it is possible to find (to within a delay of ℓ samples) a stable and causal left inverse to

$$\begin{pmatrix}\boldsymbol{A}^{-1}\boldsymbol{B} \\ -q^{-\ell-1}\mathbf{I}\end{pmatrix} \ .$$

Remark 6. Note that when a ZF GDFE exists, it can always be realized with polynomial $\boldsymbol{\mathcal{R}}(q^{-1})$ and $\boldsymbol{\mathcal{F}}(q^{-1})$. □

C. *The structure of decision feedback equalizers with asymptotically large smoothing lags*

As mentioned in Section II, it is common in theoretical investigations to let the received signal pass through a (continuous-time) filter matched to the channel, and then design the DFE to operate on this signal. In other words, the matched filter is used as an *a priori* constraint on the DFE solution. In the MIMO case, a matched filter would have to be included for every scalar channel [3], resulting in a bank of $n_y \times n_d$ matched filters. In this section, we will show that such a bank of matched filters will indeed be present in the MSE optimal GDFE, but only when the smoothing lag ℓ is allowed to go to infinity.

To prove this claim, we rewrite (13) as

$$\begin{aligned}\hat{d}(k|k+\ell) &= q^{\ell}\boldsymbol{\mathcal{R}}(q^{-1})y(k) - \boldsymbol{\mathcal{F}}(q^{-1})\tilde{d}(k-1) \\ \tilde{d}(k) &= \mathrm{f}(\hat{d}(k|k+\ell)) \ .\end{aligned}$$

We now let ℓ tend to infinity to obtain the optimum non-realizable MIMO DFE:

$$\begin{aligned}\hat{d}^{\infty}(k) &= \boldsymbol{\mathcal{R}}^{\infty}(q,q^{-1})y(k) - \boldsymbol{\mathcal{F}}^{\infty}(q^{-1})\tilde{d}(k-1) \\ \tilde{d}(k) &= \mathrm{f}(\hat{d}^{\infty}(k)) \ ,\end{aligned} \tag{26}$$

where we have defined

$$\hat{d}^{\infty}(k) \triangleq \lim_{\ell\to\infty}\hat{d}(k|k+\ell) \tag{27a}$$

$$\boldsymbol{\mathcal{R}}^{\infty}(q,q^{-1}) \triangleq \lim_{\ell\to\infty} q^{\ell}\boldsymbol{\mathcal{R}}(q^{-1}) \tag{27b}$$

$$\boldsymbol{\mathcal{F}}^{\infty}(q^{-1}) \triangleq \lim_{\ell\to\infty}\boldsymbol{\mathcal{F}}(q^{-1}) \ . \tag{27c}$$

The coefficients of the non-realizable DFE can be obtained using the following theorem:

Theorem 3: The non-causal feedforward filter $\boldsymbol{\mathcal{R}}^{\infty}(q,q^{-1})$ and the causal feedback filter $\boldsymbol{\mathcal{F}}^{\infty}(q^{-1})$ of the optimum non-realizable MIMO MMSE GDFE (26), based on correct past decisions, will be given by

$$\boldsymbol{\mathcal{R}}^{\infty}(q,q^{-1}) = \frac{1}{\rho}\tilde{\mathbf{\Gamma}}_0\mathbf{W}^{-1}\boldsymbol{\beta}_*^{-1}\tilde{\mathbf{\Gamma}}_*\boldsymbol{\tau}_*\mathbf{\Gamma}_*^{-1}\boldsymbol{M}^{-1}\boldsymbol{N} \tag{28a}$$

$$= \frac{1}{\rho}\tilde{\mathbf{\Gamma}}_0\mathbf{W}^{-1}\boldsymbol{\beta}_*^{-1}\tilde{\boldsymbol{\tau}}_*\boldsymbol{M}^{-1}\boldsymbol{N} \tag{28b}$$

$$\boldsymbol{\mathcal{F}}^{\infty}(q^{-1}) = q(\tilde{\mathbf{\Gamma}}_0\boldsymbol{\beta}\tilde{\mathbf{\Gamma}}^{-1} - \mathbf{I}) \ , \tag{28c}$$

where $\tilde{\mathbf{\Gamma}}_0$ is the leading coefficient of $\tilde{\mathbf{\Gamma}}$ and $\boldsymbol{\beta}$ is the stable and monic solution to the matrix spectral factorization

$$\boldsymbol{\beta}_*\mathbf{W}\boldsymbol{\beta} = \tilde{\mathbf{\Gamma}}_*\tilde{\mathbf{\Gamma}} + \frac{1}{\rho}\tilde{\boldsymbol{\tau}}_*\tilde{\boldsymbol{\tau}} \ . \tag{29}$$

In (29), $\mathbf{W}$ is a constant, positive definite matrix which has been introduced to make $\boldsymbol{\beta}$ monic.

Proof: See Appendix D. ■

Remark 7. Note that since $\tilde{\mathbf{\Gamma}}_0$ is non-singular, equation (29) has a unique solution. □

The filter $\boldsymbol{\tau}_*\mathbf{\Gamma}_*^{-1}\boldsymbol{M}^{-1}\boldsymbol{N}$, which appears as a right factor of (28a), can be expressed as

$$\boldsymbol{B}_*\boldsymbol{N}_*\boldsymbol{A}_*^{-1}\boldsymbol{M}_*^{-1}\boldsymbol{M}^{-1}\boldsymbol{N} = (\boldsymbol{B}_*\boldsymbol{A}_*^{-1}\boldsymbol{N}_*\boldsymbol{M}_*^{-1})\boldsymbol{M}^{-1}\boldsymbol{N} \ , \tag{30}$$

since $\boldsymbol{A}$ and $\boldsymbol{N}$ are assumed diagonal and thus commute. This filter constitutes a bank of *whitening matched filters*, where the factor $\boldsymbol{M}^{-1}\boldsymbol{N}$ performs noise whitening and the matching is performed with respect to this noise whitening filter in cascade with the channel. Hence, when the smoothing lag tends to infinity, there is no performance penalty associated with the introduction of such a filter prior to the optimization of the DFE. However, this is *not* true for the realizable DFE; no whitened matched filter is present in (19a).

D. A comparison between the DFE and MLSE

A maximum likelihood sequence estimator (MLSE) computes the transmitted signal which maximizes the conditional probability of the received signal,

$$\{\tilde{d}(k)\}_{k=1}^{N} = \arg \max_{d(k)\in\mathcal{A}} p(\{y(k)\}_{k=1}^{N} | \{d(k)\}_{k=1}^{N}) \quad (31)$$

where $d(k) \in \mathcal{A}$ implies that element i in $d(k)$ should be taken from the alphabet $\mathcal{A}_i$. When the channel and noise statistics are known, the MLSE is the optimum sequence detector.

When the channel memory is finite, the optimization in (31) can be efficiently implemented using the Viterbi algorithm [25]. Still, the complexity is very high. Assuming that the channel length is L and the size of the alphabet $\mathcal{A}_i$ is M_i, determination of the sequence $\{\tilde{d}(k)\}_{k=1}^{N}$ in (31) using the Viterbi algorithm requires on the order of

$$n_y^2 N \prod_{i=1}^{n_d} M_i^L$$

operations, which can be an enormously large number. On the other hand, calculating the coefficients of a corresponding multivariable MMSE DFE requires on the order of

$$n_y^3(\ell+1)^3 + n_y^2(\ell+1)^2 n_d$$

operations, while equalization of the N symbols requires approximately

$$N(n_y n_d(\ell+1) + n_d^2 L)$$

operations. For large n_d, alphabets and channel lengths, there is a large difference in complexity between the MLSE and the DFE. On the other hand, the difference in performance can be expected to be rather small. [6]

V. A NUMERICAL EXAMPLE

To illustrate the performance obtained by using the GDFE, a Monte Carlo simulation has been conducted. Consider the two input-two output FIR channel (cf. (8))

$$\boldsymbol{A} = \mathbf{I}, \quad \boldsymbol{B} = \begin{pmatrix} 0.979 + 0.204q^{-1} & 0.826 + 0.563q^{-1} \\ -0.843 - 0.538q^{-1} & 0.403 + 0.915q^{-1} \end{pmatrix} .$$

Over this channel, we transmit two BPSK modulated signals, that is $d_j(k) = \{+1, -1\}$, $j = 1, 2$. At the receiver, noise is added. The noise is Gaussian and will be described by the first order moving average model

$$\boldsymbol{N} = \mathbf{I}, \quad \boldsymbol{M} = \begin{pmatrix} -0.409 - 0.179q^{-1} & -0.535 + 0.717q^{-1} \\ -0.507 + 0.361q^{-1} & 0.761 - 0.181q^{-1} \end{pmatrix} .$$

This noise model has zeros in $z_{1,2} = 0.424 \pm 0.457j = 0.623e^{\pm j0.823}$. We compare the performance of two DFEs with smoothing lag $\ell = 1$:

- The MMSE GDFE, with degrees and parameters given by Theorem 1.
- The MMSE FIR DFE described in [6]. This DFE has FIR filters of degrees 2 and 1 in the feedforward and feedback links, respectively.[7]

In Fig. 4, the bit error rate (BER) is displayed as a function of the signal-to-noise ratio (SNR) of a single user. The SNR of user j is defined as

$$\text{SNR}_j \triangleq \frac{E||s_j(k)||^2}{E||w(k)||^2} \quad (32)$$

where

$$s_j(k) \triangleq \boldsymbol{A}^{-1}(q^{-1})\boldsymbol{B}_j(q^{-1})d_j(k)$$
$$w(k) \triangleq \boldsymbol{N}^{-1}(q^{-1})\boldsymbol{M}(q^{-1})v(k)$$

and $\boldsymbol{B}_j(q^{-1})$ is column j in $\boldsymbol{B}(q^{-1})$. We assume that the SNRs of the two users are identical, $\text{SNR}_1 = \text{SNR}_2$. The simulations are performed using both correct decisions and decisions from the decision device.

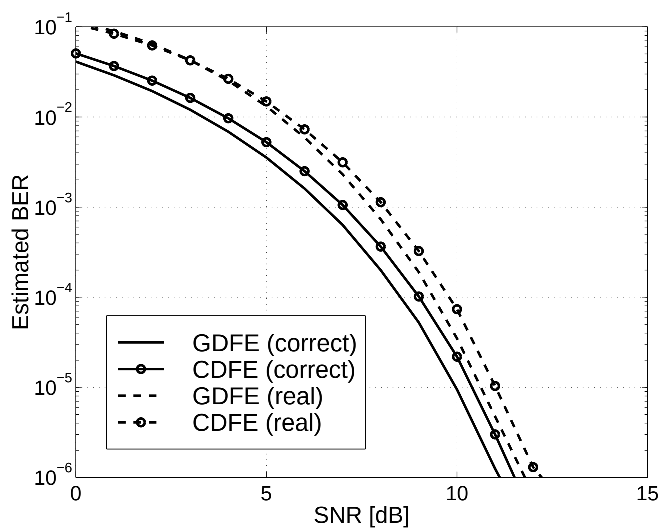

Fig. 4. The BER of the GDFE compared to the BER of the conventional DFE (CDFE) for correct decisions (solid line) and real decisions (dashed line).

From Fig. 4, we see that in some cases, it is advantageous to take the noise model into account. The GDFE does this in an optimum way, whereas the conventional DFE does not. With correct decisions, the GDFE is about 0.6 dB better than the

[6]This is true when the SISO and MISO MLSEs and DFEs are compared [26]. Although such results are not available yet, we foresee that this will be the case also for MIMO DFEs.

[7]These degrees have been chosen so that both DFEs are described using the same number of parameters.

conventional DFE over the range of investigated SNRs. With real decisions, the performance of the two DFEs is identical for low SNRs, while it differs by 0.6 dB for high SNRs.

By using higher filter degrees, the performance of the conventional DFE would be improved. In the limit, when the filter lengths go to infinity, it attains the performance of the GDFE.

From Fig. 4, we also see that with real decisions, the performance of both DFEs worsen, and that the difference between the two DFEs is smaller. This indicates that the DFE with optimal structure, and longer impulse response in the feedback filter, is more sensitive to incorrect past decisions.

For the considered FIR channel, the conventional DFE structure is optimal when the additive noise is temporally white. White noise corresponds to noise described by a moving average process, whose zeros are located in the origin. Therefore, the difference between the optimum DFE and the conventional DFE should be smaller, the closer to the origin the noise zeros are located. Conversely, when the noise zeros are located close to the unit circle, the difference should be larger.

To investigate this assumption, the locations of the zeros of the noise model are varied according to

$$z_{1,2}(r) = re^{\pm j0.823}, \begin{array}{l} r = 0.01, 0.1, 0.2, \ldots, 0.9 \\ r = 0.95, 0.98, 0.99 \end{array} ,$$

while the SNR, as defined in (32), is kept constant at 5 dB. Thus, the noise model zeros are moved along a radius, from the origin towards the unit circle. All other conditions for the simulation scenario are as in Fig. 4. The result is depicted in Fig. 5.

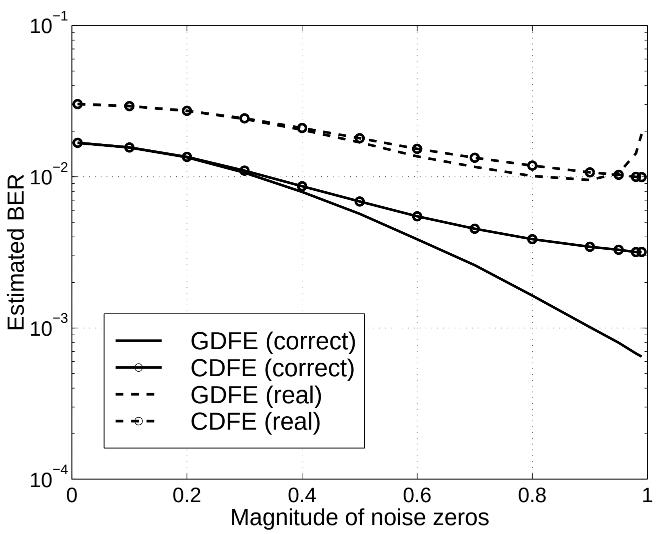

Fig. 5. The BER of the GDFE as compared to the conventional DFE (CDFE) as a function of the location of the zeros of the noise model.

The location of the noise zeros clearly affects the relative performances of the two algorithms. When the zeros are close to the origin, the performance of the two DFEs are identical, but the further out towards the unit circle the zeros are moved, the larger the difference becomes. One interesting discovery is the performance of the GDFE with real decisions when the noise zeros are located very close to the unit circle: In this scenario, error propagation causes very bad performance for the GDFE. The reason is that when the noise zeros are close to the unit circle, so are the poles of the feedback filter. The impulse response of the feedback filter then becomes very long, leading to longer error bursts.

VI. Conclusions

From a practical point of view, a decision feedback equalizer must be realizable. Also, optimum performance can be achieved only if the structure of the DFE is appropriate for the considered scenario. We have presented a generalized DFE with optimal structure, derived under the constraint of realizability. The IIR filters in this DFE are obtained from closed form design equations, which involve the channel and noise descriptions. By allowing the smoothing lag to go to infinity, we have also derived the optimum non-realizable DFE.

From knowledge of the optimum DFE structure, it is also possible to give guidelines on how to choose the filter degrees in a conventional DFE.

The performance of the general DFE is demonstrated in a numerical example. This example indicates that for heavily colored noise, a GDFE with IIR filters outperforms the conventional FIR DFE. However, the conventional DFE seems to be less sensitive to the presence of incorrect past decisions: When incorrect decisions occur, the difference in performance is reduced.

Since the GDFE seems to be sensitive to incorrect decisions, robust design which take the signal uncertainty into account would be of interest. Such methods exist for scalar DFEs, see [21] and they can be generalized to MIMO systems [27].

The performance of the GDFE as compared to the conventional DFE when identified models are employed is a topic for future research. Particular attention should be paid to the noise model: Since autoregressive models are easier to obtain than MA models, it could be of interest to approximate an MA noise process with an AR description [28], [26]. This would have the additional advantage that the problem with error propagation would be reduced since the conventional FIR DFE is optimum for this model structure.

Acknowledgments

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Appendix

A. Proof of Theorem 1

Consider the channel described by (8), and the general multivariable decision feedback equalizer (13). Insert the expression for the symbol estimate $\hat{d}(k-\ell|k)$ into the expression (15) for the estimation error and assume correct past decisions:

$$\begin{aligned}\varepsilon(k-\ell) &= d(k-\ell) - \hat{d}(k-\ell|k) \\ &= d(k-\ell) - \boldsymbol{\mathcal{R}} y(k) + \boldsymbol{\mathcal{F}} d(k-\ell-1) \,. \end{aligned} \tag{33}$$

Insert $y(k)$ from (8) into (33) and rearrange:

$$\begin{aligned}\varepsilon(k-\ell) &= d(k-\ell) - \boldsymbol{\mathcal{R}}(\boldsymbol{A}^{-1}\boldsymbol{B}d(k) + \boldsymbol{N}^{-1}\boldsymbol{M}v(k)) \\ &\quad + \boldsymbol{\mathcal{F}}d(k-\ell-1) \\ &= (q^{-\ell}\mathbf{I} - \boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} + q^{-\ell-1}\boldsymbol{\mathcal{F}})d(k) \\ &\quad - \boldsymbol{\mathcal{R}}\boldsymbol{N}^{-1}\boldsymbol{M}v(k) \,. \end{aligned} \tag{34}$$

Introduce the alternative estimate $\hat{d}_a(k-\ell|k) \stackrel{\triangle}{=} \hat{d}(k-\ell|k) + n(k)$, where the variation $n(k)$ is a linear function of all signals which the estimate $\hat{d}(k-\ell|k)$ may be based upon. Thus,

$$n(k) \stackrel{\triangle}{=} n_1(k) + n_2(k)$$

where

$$n_1(k) \stackrel{\triangle}{=} \boldsymbol{\mathcal{G}}_1 y(k) \tag{35a}$$
$$n_2(k) \stackrel{\triangle}{=} \boldsymbol{\mathcal{G}}_2 d(k-\ell-1) \,. \tag{35b}$$

Above, $\boldsymbol{\mathcal{G}}_1(q^{-1})$ and $\boldsymbol{\mathcal{G}}_2(q^{-1})$ are arbitrary stable and causal rational matrices. If the estimation error obtained with (13) is orthogonal to any admissible variation (35a), (35b), that is,

$$E\varepsilon(k-\ell)n_1^H(k) = 0 \tag{36a}$$
$$E\varepsilon(k-\ell)n_2^H(k) = 0 \tag{36b}$$

then $\hat{d}_a(k-\ell|k) = \hat{d}(k-\ell|k)$ or equivalently $n(k) \equiv 0$ minimizes the estimation error covariance matrix (14): For any estimation error covariance matrix $\tilde{\mathbf{P}}$ obtained with $n(k) \neq 0$, $\tilde{\mathbf{P}} - \mathbf{P}$ will be positive definite. We must thus assure that (36a) and (36b) are fulfilled.

To compute the cross-correlations (36a) and (36b), we will use Parseval's relation for complex signals [29] and evaluate the expressions in the frequency domain. We thus insert (34) and (35a) into (36a) and use Parseval's relation to obtain

$$\begin{aligned}E\varepsilon(k-\ell)n_1^H(k) = \frac{\lambda_d}{2\pi j}\oint \{(z^{-\ell}\mathbf{I} - \boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} + z^{-\ell-1}\boldsymbol{\mathcal{F}}) \times \\ \boldsymbol{B}_*\boldsymbol{A}_*^{-1} - \rho\boldsymbol{\mathcal{R}}\boldsymbol{N}^{-1}\boldsymbol{M}\boldsymbol{M}_*\boldsymbol{N}_*^{-1}\}\,\boldsymbol{\mathcal{G}}_{1*}\frac{dz}{z} \,. \end{aligned} \tag{37}$$

For an explanation of how Parseval's relation is used to obtain this expression, see [30]. Since $\boldsymbol{A}$ and $\boldsymbol{N}$ are both diagonal, we can express (37) as

$$\begin{aligned}E\varepsilon(k-\ell)n_1^H(k) = \frac{\lambda_d}{2\pi j}\oint \{(z^{-\ell}\mathbf{I} - \boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} + z^{-\ell-1}\boldsymbol{\mathcal{F}}) \times \\ \boldsymbol{B}_*\boldsymbol{N}_* - \rho\boldsymbol{\mathcal{R}}\boldsymbol{N}^{-1}\boldsymbol{M}\boldsymbol{M}_*\boldsymbol{A}_*\}\,\boldsymbol{A}_*^{-1}\boldsymbol{N}_*^{-1}\boldsymbol{\mathcal{G}}_{1*}\frac{dz}{z} \,. \end{aligned} \tag{38}$$

From (38) we see that $E\varepsilon(k-\ell)n_1^H(k) = 0$ if the integrand is analytic inside the integration path $|z| = 1$. To assure that the integral vanishes for *any* stable and causal $\boldsymbol{\mathcal{G}}_1$, this condition is also necessary, see Lemma A.1 in [24]. According to the assumption (9), $\boldsymbol{A}^{-1}$ and $\boldsymbol{N}^{-1}$ are stable. This implies that $\boldsymbol{A}_*^{-1}$ and $\boldsymbol{N}_*^{-1}$ are analytic inside the unit circle. The same applies for $\boldsymbol{\mathcal{G}}_{1*}$, since $\boldsymbol{\mathcal{G}}_1$ is required to be stable. If

$$\begin{aligned}(z^{-\ell}\mathbf{I} - \boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} + z^{-\ell-1}\boldsymbol{\mathcal{F}})\boldsymbol{B}_*\boldsymbol{N}_* \\ - \rho\boldsymbol{\mathcal{R}}\boldsymbol{N}^{-1}\boldsymbol{M}\boldsymbol{M}_*\boldsymbol{A}_* = z\boldsymbol{\mathcal{L}}_{2*} \end{aligned} \tag{39}$$

for some rational matrix $\boldsymbol{\mathcal{L}}_{2*}(z)$ with all its poles *outside* the unit circle, then the integrand will be analytic inside the unit circle. Proceeding in the same way with (36b) results in a second condition

$$z^{-\ell}\mathbf{I} - \boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} + z^{-\ell-1}\boldsymbol{\mathcal{F}} = z^{-\ell}\boldsymbol{\mathcal{L}}_{1*} \tag{40}$$

for some rational matrix $\boldsymbol{\mathcal{L}}_{1*}(z)$ with all its poles *outside* the unit circle. However, none of the terms on the left hand side of (40) can have poles outside the unit circle. Therefore, we conclude that $\boldsymbol{\mathcal{L}}_{1*}(z)$ must be a polynomial matrix. Thus,

$$z^{-\ell}\mathbf{I} - \boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} + z^{-\ell-1}\boldsymbol{\mathcal{F}} = z^{-\ell}\boldsymbol{L}_{1*} \tag{41}$$

for some polynomial matrix $\boldsymbol{L}_{1*}(z)$. We can now insert (41) into (39) to obtain

$$z^{-\ell}\boldsymbol{L}_{1*}\boldsymbol{B}_*\boldsymbol{N}_* - \rho\boldsymbol{\mathcal{R}}\boldsymbol{N}^{-1}\boldsymbol{M}\boldsymbol{M}_*\boldsymbol{A}_* = z\boldsymbol{\mathcal{L}}_{2*} \,. \tag{42}$$

We realize that neither of the terms on the left hand side of (42) can have any poles outside the unit circle. Therefore, $\boldsymbol{\mathcal{L}}_{2*}(z)$ cannot have any such poles either, and we conclude that $\boldsymbol{\mathcal{L}}_{2*}(z)$ must be a polynomial, rather than a rational, matrix. Hence we can express equation (42) as

$$z^{-\ell}\boldsymbol{L}_{1*}\boldsymbol{B}_*\boldsymbol{N}_* - \rho\boldsymbol{\mathcal{R}}\boldsymbol{N}^{-1}\boldsymbol{M}\boldsymbol{M}_*\boldsymbol{A}_* = z\boldsymbol{L}_{2*} \,. \tag{43}$$

for some polynomial matrix $\boldsymbol{L}_{2*}(z)$. In the integrand (43), the poles contributed by $\boldsymbol{N}^{-1}(z^{-1})$ must be canceled by a corresponding factor in $\boldsymbol{\mathcal{R}}(z^{-1})$. Also, the polynomial matrix $\boldsymbol{M}(z^{-1})$ contributes poles in the origin, which must be canceled by a corresponding factor in $\boldsymbol{\mathcal{R}}(z^{-1})$. We therefore insert

$$\boldsymbol{\mathcal{R}} = \boldsymbol{S}\boldsymbol{M}^{-1}\boldsymbol{N} \tag{44}$$

where $\boldsymbol{S}$ is an arbitrary polynomial matrix, into (41) and (43) and rearrange:

$$\begin{aligned}\mathbf{I} - z^{\ell}\boldsymbol{S}\boldsymbol{M}^{-1}\boldsymbol{N}\boldsymbol{A}^{-1}\boldsymbol{B} + z^{-1}\boldsymbol{\mathcal{F}} &= \boldsymbol{L}_{1*} \\ z^{-\ell}\boldsymbol{L}_{1*}\boldsymbol{B}_*\boldsymbol{N}_* - \rho\boldsymbol{S}\boldsymbol{M}_*\boldsymbol{A}_* &= z\boldsymbol{L}_{2*} \,. \end{aligned} \tag{45}$$

We now use that $\boldsymbol{A}^{-1}$ and $\boldsymbol{N}$ are diagonal and hence commute. We also insert (17a) and (17b) into (45):

$$\begin{aligned}\mathbf{I} - z^{\ell}\boldsymbol{S}\boldsymbol{\Gamma}^{-1}\boldsymbol{\tau} + z^{-1}\boldsymbol{\mathcal{F}} &= \boldsymbol{L}_{1*} \\ z^{-\ell}\boldsymbol{L}_{1*}\boldsymbol{\tau}_* - \rho\boldsymbol{S}\boldsymbol{\Gamma}_* &= z\boldsymbol{L}_{2*} \,. \end{aligned} \tag{46}$$

Equation (46) can be further simplified by using the equation (18) and multiplying with $\tilde{\boldsymbol{\Gamma}}(q^{-1})$ from the right:

$$\tilde{\boldsymbol{\Gamma}} - z^{\ell}\boldsymbol{S}\tilde{\boldsymbol{\tau}} + z^{-1}\boldsymbol{\mathcal{F}}\tilde{\boldsymbol{\Gamma}} = \boldsymbol{L}_{1*}\tilde{\boldsymbol{\Gamma}} \tag{47a}$$
$$z^{-\ell}\boldsymbol{L}_{1*}\boldsymbol{\tau}_* - \rho\boldsymbol{S}\boldsymbol{\Gamma}_* = z\boldsymbol{L}_{2*} \,. \tag{47b}$$

Since $\mathcal{F}$ is the only remaining rational matrix in (47a), its poles must be canceled by a corresponding factor in $\tilde{\mathbf{\Gamma}}$. We thus conclude that

$$\mathcal{F} = \boldsymbol{Q}\tilde{\mathbf{\Gamma}}^{-1}$$

where $\boldsymbol{Q}$ is an undetermined polynomial matrix. Note that $\boldsymbol{Q}$ and $\tilde{\mathbf{\Gamma}}$ may have common factors. We can now insert this expression into (47a) to obtain

$$\tilde{\mathbf{\Gamma}} - z^{\ell}\boldsymbol{S}\tilde{\boldsymbol{\tau}} + z^{-1}\boldsymbol{Q} = \boldsymbol{L}_{1*}\tilde{\mathbf{\Gamma}} \tag{48a}$$
$$z^{-\ell}\boldsymbol{L}_{1*}\boldsymbol{\tau}_* - \rho\boldsymbol{S}\mathbf{\Gamma}_* = z\boldsymbol{L}_{2*}\,. \tag{48b}$$

By exchanging the unknown z for q, equation (48a) coincides with (20a), whereas (48b) coincides with (20b).

The Diophantine equations (48a) and (48b) are *double sided*, that is, they contain powers of both z^{-1} and z. Thus, both the powers of z^{-1} and z on the left hand side must match the corresponding powers on the right hand side. For this purpose, we list the degrees in z^{-1} and z for each term in (48a) and (48b):
Equation (48a):

$$z^{-1}: \quad \delta\tilde{\Gamma},\ \delta S + \delta\tilde{\tau} - \ell,\ \delta Q + 1,\ \delta\tilde{\Gamma} \tag{49a}$$
$$z: \quad 0,\ \ell,\ 0,\ \delta L_1 \tag{49b}$$

Equation (48b):

$$z^{-1}: \quad \ell,\ \delta S,\ 0 \tag{49c}$$
$$z: \quad \delta L_1 + \delta\tau - \ell,\ \delta\Gamma,\ \delta L_2 + 1 \tag{49d}$$

From (49b) and (49c) we immediately obtain the conditions for the degrees of $\boldsymbol{L}_1$ and $\boldsymbol{S}$ respectively:

$$\delta L_1 = \ell; \quad \delta S = \ell\,.$$

If we insert $\delta S = \ell$ into (49a), we obtain (21b). Finally, by inserting $\delta L_1 = \ell$ into (49d), we obtain (21d).

We will now show that equations (48a) and (48b) have a solution with the degrees specified by (21a)–(21d). For this purpose, we rewrite (48a) and (48b) as two systems of linear equations. Two matrix polynomials are identical if and only if all the corresponding coefficient matrices are identical. We must thus adjust the coefficients of $\boldsymbol{S}$, $\boldsymbol{Q}$, $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ so that the expressions for the matrix coefficients for each power of z and z^{-1} are equal on the left and right hand side of (48a) and (48b). We thus evaluate the expressions for the matrix coefficients, conjugate, transpose and equate the left and right hand sides. For (48a) we obtain

$$\begin{pmatrix} \tilde{\boldsymbol{\tau}}_0^H & & 0 & \tilde{\mathbf{\Gamma}}_0^H & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ \tilde{\boldsymbol{\tau}}_{\delta\alpha}^H & & \tilde{\boldsymbol{\tau}}_0^H & \tilde{\mathbf{\Gamma}}_{\delta\alpha}^H & & \tilde{\mathbf{\Gamma}}_0^H \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & & \tilde{\boldsymbol{\tau}}_{\delta\alpha}^H & 0 & & \tilde{\mathbf{\Gamma}}_{\delta\alpha}^H \end{pmatrix} \begin{pmatrix} \mathbf{S}_0^H \\ \vdots \\ \mathbf{S}_\ell^H \\ \mathbf{L}_{1\ell} \\ \vdots \\ \mathbf{L}_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \tilde{\mathbf{\Gamma}}_0^H \\ \boldsymbol{\alpha}_1^H \\ \vdots \\ \boldsymbol{\alpha}_{\delta\alpha}^H \end{pmatrix} \tag{50}$$

where we have defined

$$\boldsymbol{\alpha}(z^{-1}) = \tilde{\mathbf{\Gamma}}(z^{-1}) + z^{-1}\boldsymbol{Q}(z^{-1})\,. \tag{51}$$

Note that $\delta\alpha = \max(\delta\tilde{\tau}, \delta\tilde{\Gamma})$. In (50) $\tilde{\boldsymbol{\tau}}_m = 0$ if $m > \delta\tilde{\tau}$ and $\tilde{\mathbf{\Gamma}}_m = 0$ if $m > \delta\tilde{\Gamma}$. Proceeding in the same manner with (48b) results in

$$\begin{pmatrix} -\rho\mathbf{\Gamma}_{\delta L_2+1} & & 0 & \boldsymbol{\tau}_{\delta L_2+1} & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ -\rho\mathbf{\Gamma}_0 & & -\rho\mathbf{\Gamma}_{\delta L_2+1} & \boldsymbol{\tau}_0 & & \boldsymbol{\tau}_{\delta L_2+1} \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & \dots & -\rho\mathbf{\Gamma}_0 & 0 & \dots & \boldsymbol{\tau}_0 \end{pmatrix} \times$$
$$\times \begin{pmatrix} \mathbf{S}_0^H \\ \vdots \\ \mathbf{S}_\ell^H \\ \mathbf{L}_{1\ell} \\ \vdots \\ \mathbf{L}_{10} \end{pmatrix} = \begin{pmatrix} \mathbf{L}_{2\delta L_2} \\ \vdots \\ \mathbf{L}_{20} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{52}$$

where $\mathbf{\Gamma}_m = 0$ if $m > \delta\Gamma$ and $\boldsymbol{\tau}_m = 0$ if $m > \delta\tau$.

Neither (50) nor (52) can be solved directly, since unknown coefficient matrices appear on their right hand sides. However, we can combine the first $(\ell+1)n_d$ equations from (50) with the last $(\ell+1)n_y$ equations from (52) to obtain a system with equal number of equations and unknowns

$$\begin{pmatrix} \tilde{\boldsymbol{\tau}}_0^H & & 0 & \tilde{\mathbf{\Gamma}}_0^H & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ \tilde{\boldsymbol{\tau}}_\ell^H & \dots & \tilde{\boldsymbol{\tau}}_0^H & \tilde{\mathbf{\Gamma}}_\ell^H & \dots & \tilde{\mathbf{\Gamma}}_0^H \\ -\rho\mathbf{\Gamma}_0 & \dots & -\rho\mathbf{\Gamma}_\ell & \boldsymbol{\tau}_0 & & \boldsymbol{\tau}_\ell \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & \dots & -\rho\mathbf{\Gamma}_0 & 0 & \dots & \boldsymbol{\tau}_0 \end{pmatrix} \begin{pmatrix} \mathbf{S}_0^H \\ \vdots \\ \mathbf{S}_\ell^H \\ \mathbf{L}_{1\ell} \\ \vdots \\ \mathbf{L}_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \tilde{\mathbf{\Gamma}}_0^H \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{53}$$

where only known coefficient matrices appear on the right hand side. We will now show that this system of linear equations has a unique solution whenever $\rho > 0$. Define

$$\mathbf{T} \triangleq \begin{pmatrix} \boldsymbol{\tau}_0 & \dots & \boldsymbol{\tau}_\ell \\ & \ddots & \vdots \\ 0 & \dots & \boldsymbol{\tau}_0 \end{pmatrix};\ \tilde{\mathbf{T}} \triangleq \begin{pmatrix} \tilde{\boldsymbol{\tau}}_0 & \dots & \tilde{\boldsymbol{\tau}}_\ell \\ & \ddots & \vdots \\ 0 & \dots & \tilde{\boldsymbol{\tau}}_0 \end{pmatrix}$$
$$\mathbf{G} \triangleq \begin{pmatrix} \mathbf{\Gamma}_0 & \dots & \mathbf{\Gamma}_\ell \\ & \ddots & \vdots \\ 0 & \dots & \mathbf{\Gamma}_0 \end{pmatrix};\ \tilde{\mathbf{G}} \triangleq \begin{pmatrix} \tilde{\mathbf{\Gamma}}_0 & \dots & \tilde{\mathbf{\Gamma}}_\ell \\ & \ddots & \vdots \\ 0 & \dots & \tilde{\mathbf{\Gamma}}_0 \end{pmatrix}$$

$$\theta \triangleq \begin{pmatrix} \mathbf{S}_0 & \dots & \mathbf{S}_\ell & \mathbf{L}_{1\ell}^H & \dots & \mathbf{L}_{10}^H \end{pmatrix}^H$$
$$\tilde{\mathbf{g}} \triangleq \begin{pmatrix} 0 & \dots & \tilde{\mathbf{\Gamma}}_0 & 0 & \dots & 0 \end{pmatrix}^H\,.$$

Equation (53) can then be written as

$$\underbrace{\begin{pmatrix} \tilde{\mathbf{T}}^H & \tilde{\mathbf{G}}^H \\ -\rho\mathbf{G} & \mathbf{T} \end{pmatrix}}_{\mathbf{V}} \theta = \tilde{\mathbf{g}}\,.$$

The matrix $\mathbf{V}$ is non-singular since

$$\begin{aligned}\det \mathbf{V} &= (-1)^r \det \tilde{\mathbf{G}}^H \det\left(\rho\mathbf{G} + \mathbf{T}\tilde{\mathbf{G}}^{-H}\tilde{\mathbf{T}}^H\right) \\ &= (-1)^r \det \tilde{\mathbf{G}}^H \det\left(\rho\mathbf{G} + \mathbf{T}(\mathbf{G}^{-1}\mathbf{T})^H\right) \\ &= (-1)^r \det \tilde{\mathbf{G}}^H \det\left(\rho\mathbf{G}\mathbf{G}^H + \mathbf{T}\mathbf{T}^H\right) \det \mathbf{G}^{-H}\end{aligned}$$

where $(\cdot)^{-H} = ((\cdot)^{-1})^H$. Above, the integer r compensates for the sign shifts caused by the elementary operations performed on the determinant. In the second equality, we have used that $\tilde{\mathbf{T}}\tilde{\mathbf{G}}^{-1} = \mathbf{G}^{-1}\mathbf{T}$, which is a consequence of (18). Now, since both $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are non-singular, so are $\mathbf{G}^{-H}$ and $\tilde{\mathbf{G}}^H$. Also, $\mathbf{G}\mathbf{G}^H$ is positive definite. Therefore, the matrix $\mathbf{V}$ is non-singular whenever $\rho > 0$.

After having solved (53) for $\mathbf{S}_m^H$ and $\mathbf{L}_{1m}$, we use (50) to calculate the coefficients of $\boldsymbol{\alpha}$. We then compute the feedback filter $\boldsymbol{Q}$ with the aid of (51):

$$\boldsymbol{Q}(z^{-1}) = z(\boldsymbol{\alpha}(z^{-1}) - \tilde{\boldsymbol{\Gamma}}(z^{-1})) \ . \tag{54}$$

Since the solution to (53) is unique, and $\boldsymbol{\alpha}(z^{-1})$ and $\boldsymbol{Q}(z^{-1})$ are determined explicitly from (50) and (54), we conclude that there exists a unique solution to (48a) and (48b).

It remains to derive the expression for the residual MSE at the output of the estimator. Replace z^{-1} and z with q^{-1} and q respectively in (41) and (44) and use the resulting relation to rewrite (33) as:

$$\varepsilon(k-\ell) = q^{-\ell}\boldsymbol{L}_{1*}(q)d(k) - \boldsymbol{S}(q^{-1})v(k) \ . \tag{55}$$

Since $d(k-n)$ and $v(k-m)$ are white and mutually uncorrelated for all n and m, we easily obtain

$$\mathbf{P} \stackrel{\triangle}{=} E\varepsilon(k)\varepsilon^H(k) = \lambda_d \sum_{n=0}^{\ell} \mathbf{L}_{1n}^H \mathbf{L}_{1n} + \lambda_v \sum_{n=0}^{\ell} \mathbf{S}_n \mathbf{S}_n^H \ ,$$

which coincides with (22).

B. Proof of Theorem 2

We start with the expression (13), and use the assumption on correct past symbols:

$$\hat{d}(k-\ell|k) = \boldsymbol{\mathcal{R}}(q^{-1})y(k) - \boldsymbol{\mathcal{F}}(q^{-1})d(k-\ell-1) \ .$$

We then insert $y(k)$ from (8) and rearrange:

$$\begin{aligned}\hat{d}(k-\ell|k) &= (\boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} - q^{-\ell-1}\boldsymbol{\mathcal{F}})d(k) + \boldsymbol{\mathcal{R}}\boldsymbol{N}^{-1}\boldsymbol{M}v(k) \\ &= d(k-\ell) + (\boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} - q^{-\ell-1}\boldsymbol{\mathcal{F}} - q^{-\ell}\mathbf{I})d(k) \\ &\quad + \boldsymbol{\mathcal{R}}\boldsymbol{N}^{-1}\boldsymbol{M}v(k) \ . \end{aligned} \tag{56}$$

Since $d(k)$ and $v(m)$ are uncorrelated for all k and m, the zero-forcing condition (16) can be satisfied if and only if

$$(\boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{B} - q^{-\ell-1}\boldsymbol{\mathcal{F}} - q^{-\ell}\mathbf{I}) = 0 \ ,$$

which coincides with (23).

C. Proof of Lemma 1

We will begin by showing necessity of the conditions in Lemma 1. In (23), we substitute the delay operator q^{-1} for the complex variable λ:

$$\lambda^{\ell}\mathbf{I} = \boldsymbol{\mathcal{R}}(\lambda)\boldsymbol{A}^{-1}(\lambda)\boldsymbol{B}(\lambda) - \lambda^{\ell+1}\boldsymbol{\mathcal{F}}(\lambda) \ . \tag{57}$$

Studying the zeros of $\boldsymbol{B}(z^{-1})$ at infinity will then be equivalent to studying the zeros of $\boldsymbol{B}(\lambda)$ at $\lambda = 0$.

The matrix $\boldsymbol{B}(\lambda)$ can be written as

$$\boldsymbol{B}(\lambda) = \boldsymbol{U}(\lambda)\boldsymbol{D}(\lambda)\boldsymbol{V}(\lambda) \ , \tag{58}$$

where $\boldsymbol{U}(\lambda)$ and $\boldsymbol{V}(\lambda)$ are unimodular matrices and $\boldsymbol{D}(\lambda)$ is the Smith form of $\boldsymbol{B}(\lambda)$. By extracting the zeros at $\lambda = 0$, $\boldsymbol{D}(\lambda)$ may be decomposed as

$$\boldsymbol{D}(\lambda) = \boldsymbol{D}_1(\lambda)\boldsymbol{D}_2(\lambda) \ , \tag{59}$$

where

$$\boldsymbol{D}_2(\lambda) \stackrel{\triangle}{=} \operatorname{diag}\begin{pmatrix}\lambda^{\mu_1} & \lambda^{\mu_2} & \ldots & \lambda^{\mu_r} & 0 & \ldots & 0\end{pmatrix}$$

and $r = \operatorname{rank} \boldsymbol{B}(\lambda)$. We now insert (58) and (59) into (57) and rearrange:

$$\boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{U}\boldsymbol{D}_1\boldsymbol{D}_2 + \boldsymbol{\mathcal{F}}\boldsymbol{V}^{-1}\lambda^{\ell+1} = \boldsymbol{V}^{-1}\lambda^{\ell} \ , \tag{60}$$

where we have dropped the polynomial argument λ to simplify the notation. Now, if $r < n_d$, then

$$\boldsymbol{\Delta}_1 = \operatorname{diag}\begin{pmatrix}\underbrace{1 \ \ \ldots \ \ 1}_{n_d-1} & \lambda^{\ell+1}\end{pmatrix}$$

is a right factor of $\boldsymbol{A}^{-1}\boldsymbol{U}\boldsymbol{D}_1\boldsymbol{D}_2$ and $\boldsymbol{V}^{-1}\lambda^{\ell+1}$. Since $\boldsymbol{\Delta}_1$ is not a right factor of $\boldsymbol{V}^{-1}\lambda^{\ell}$, equation (60) will not have any solution, according to the theory of Diophantine equations, see [23]. Since $r \leq n_d$, we conclude that $r = n_d$ is a necessary condition for the existence of a solution to (23). Note that since $\boldsymbol{B}$ can have rank n_d only when $n_y \geq n_d$, the rank condition implies that $n_y \geq n_d$.

Now, assume that $\mu_n \geq \ell + 1$ for some n. In this case,

$$\boldsymbol{\Delta}_2 = \operatorname{diag}\begin{pmatrix}\underbrace{1 \ \ \ldots \ \ 1}_{n-1} & \lambda^{\ell+1} & \underbrace{1 \ \ \ldots \ \ 1}_{n_d-n}\end{pmatrix}$$

will be a right factor of $\boldsymbol{A}^{-1}\boldsymbol{U}\boldsymbol{D}_1\boldsymbol{D}_2$ and $\boldsymbol{V}^{-1}\lambda^{\ell+1}$, but not of $\boldsymbol{V}^{-1}\lambda^{\ell}$. Hence the Diophantine equation (60) will lack a solution. We can therefore conclude that $\mu_n \leq \ell \ \forall \ n$ is a necessary condition for the existence of a solution to (60). This completes the proof of the necessity.

We will now proove sufficiency. When $\mu_n \leq \ell \ \forall \ n$, the following factorizations are possible:

$$\begin{aligned}\lambda^{\ell+1}\mathbf{I} &= \boldsymbol{I}_1\boldsymbol{D}_2 \\ \lambda^{\ell}\mathbf{I} &= \boldsymbol{I}_2\boldsymbol{D}_2\end{aligned}$$

where $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are stable and causal rational matrices. These factorizations can be inserted into (60) to yield

$$\boldsymbol{\mathcal{R}}\boldsymbol{A}^{-1}\boldsymbol{U}\boldsymbol{D}_1 + \boldsymbol{\mathcal{F}}\boldsymbol{V}^{-1}\boldsymbol{I}_1 = \boldsymbol{V}^{-1}\boldsymbol{I}_2 \ . \tag{61}$$

We now observe that $\boldsymbol{A}^{-1}\boldsymbol{U}\boldsymbol{D}_1$ and $\boldsymbol{V}^{-1}\boldsymbol{I}_1$ are right coprime, since

$$\begin{pmatrix} \boldsymbol{A}^{-1}\boldsymbol{U}\boldsymbol{D}_1 \\ \boldsymbol{V}^{-1}\boldsymbol{I}_1 \end{pmatrix}$$

has full rank n_d for all λ. This can be realized by noting that

- $\boldsymbol{A}^{-1}\boldsymbol{U}\boldsymbol{D}_1$ has full rank n_d for $\lambda = 0$.
- $\boldsymbol{V}^{-1}\boldsymbol{I}_1$ has full rank n_d for $\lambda \neq 0$.

Since $\boldsymbol{A}^{-1}\boldsymbol{U}\boldsymbol{D}_1$ and $\boldsymbol{V}^{-1}\boldsymbol{I}_1$ are right coprime, equation (61) will always have a solution. This completes the proof also of the sufficiency.

D. Proof of Theorem 3

Consider the GDFE (13) with correct past decision:

$$\hat{d}(k|k+\ell) = \boldsymbol{\mathcal{R}}(q^{-1})y(k+\ell) - \boldsymbol{\mathcal{F}}(q^{-1})d(k-1)\,. \quad (62)$$

If we now let the smoothing lag tend to infinity, we can equivalently express the symbol estimate of the asymptotic GDFE as

$$\hat{d}^{\infty}(k) = \boldsymbol{\mathcal{R}}^{\infty}(q,q^{-1})y(k) - \boldsymbol{\mathcal{F}}^{\infty}(q^{-1})d(k-1)\,. \quad (63)$$

where we have defined $\hat{d}^{\infty}(k)$ and $\boldsymbol{\mathcal{R}}^{\infty}(q,q^{-1})$ in (27a) and (27b), respectively. Note that $\boldsymbol{\mathcal{R}}^{\infty}(q,q^{-1})$ is stable but noncausal.

By using the channel model (8) the estimation error can be expressed as

$$\begin{aligned} \varepsilon(k) &= d(k) - \hat{d}^{\infty}(k) \\ &= (\mathbf{I} - \boldsymbol{\mathcal{R}}^{\infty}(q,q^{-1})\boldsymbol{A}^{-1}\boldsymbol{B} + q^{-1}\boldsymbol{\mathcal{F}}^{\infty})d(k) \\ &- \boldsymbol{\mathcal{R}}^{\infty}(q,q^{-1})\boldsymbol{N}^{-1}\boldsymbol{M}v(k) \end{aligned}$$

Introduce the alternate estimate $\hat{d}_a^{\infty}(k)$

$$\hat{d}_a^{\infty}(k) \stackrel{\triangle}{=} \hat{d}^{\infty}(k) + n(k)$$

where the variation $n(k)$ is based on all signals the estimate $\hat{d}^{\infty}(k)$ may be based upon:

$$n(k) \stackrel{\triangle}{=} n_1(k) + n_2(k)$$

with

$$n_1(k) \stackrel{\triangle}{=} \boldsymbol{\mathcal{G}}_1^{\infty}(q,q^{-1})y(k) \quad (64a)$$
$$n_2(k) \stackrel{\triangle}{=} \boldsymbol{\mathcal{G}}_2^{\infty}(q^{-1})d(k-1) \quad (64b)$$

and where $\boldsymbol{\mathcal{G}}_1^{\infty}(q,q^{-1})$ and $\boldsymbol{\mathcal{G}}_2^{\infty}(q^{-1})$ are arbitrary stable rational matrices. In addition, $\boldsymbol{\mathcal{G}}_2^{\infty}(q^{-1})$ is required to be causal. If the estimation error obtained with (63) is orthogonal to any admissible variation (64a), (64b), that is if

$$E\varepsilon(k)n_1^H(k) = 0 \quad (65a)$$
$$E\varepsilon(k)n_2^H(k) = 0 \quad (65b)$$

then $\hat{d}_a^{\infty}(k) = \hat{d}^{\infty}(k)$ or equivalently $n(k) \equiv 0$ minimizes the MSE (14). We must thus assure that (65a) and (65b) are fulfilled.

To derive the design equations of the MSE optimum GDFE with asymptotically large smoothing lag, we use Parseval's relation to evaluate the cross-correlation (65a) in the frequency domain. Proceeding as in Appendix A, we obtain

$$\begin{aligned} E\varepsilon(k)n_1^H(k) = \frac{\lambda_d}{2\pi j}\oint \big\{ (\mathbf{I} - \boldsymbol{\mathcal{R}}^{\infty}\boldsymbol{A}^{-1}\boldsymbol{B} + z^{-1}\boldsymbol{\mathcal{F}}^{\infty}) \times \\ \boldsymbol{B}_*\boldsymbol{N}_* - \rho\boldsymbol{\mathcal{R}}^{\infty}\boldsymbol{N}^{-1}\boldsymbol{M}\boldsymbol{M}_*\boldsymbol{A}_* \big\}\, \boldsymbol{A}_*^{-1}\boldsymbol{N}_*^{-1}\boldsymbol{\mathcal{G}}_{1*}^{\infty}\frac{dz}{z}\,. \quad (66) \end{aligned}$$

Since $\boldsymbol{\mathcal{G}}_{1*}^{\infty}$ contributes poles in the origin, this integral will equal zero if and only if

$$\begin{aligned} (\mathbf{I} - \boldsymbol{\mathcal{R}}^{\infty}\boldsymbol{A}^{-1}\boldsymbol{B} + z^{-1}\boldsymbol{\mathcal{F}}^{\infty})\boldsymbol{B}_*\boldsymbol{N}_* - \\ \rho\boldsymbol{\mathcal{R}}^{\infty}\boldsymbol{N}^{-1}\boldsymbol{M}\boldsymbol{M}_*\boldsymbol{A}_* = 0\,. \quad (67) \end{aligned}$$

This is proven formally in Lemma A.1 in [24].

Using the same argument as in Appendix A, the cross-correlation in (65b) can be simplified to yield

$$\mathbf{I} - \boldsymbol{\mathcal{R}}^{\infty}\boldsymbol{A}^{-1}\boldsymbol{B} + z^{-1}\boldsymbol{\mathcal{F}}^{\infty} = \boldsymbol{\mathcal{L}}_{1*} \quad (68)$$

for some stable and causal rational matrix $\boldsymbol{\mathcal{L}}_{1*}$. Inserting (68) into (67) yields:

$$\boldsymbol{\mathcal{L}}_{1*}\boldsymbol{B}_*\boldsymbol{N}_* - \rho\boldsymbol{\mathcal{R}}^{\infty}\boldsymbol{N}^{-1}\boldsymbol{M}\boldsymbol{M}_*\boldsymbol{A}_* = 0\,.$$

We can now express the feedforward filter as

$$\boldsymbol{\mathcal{R}}^{\infty} = \frac{1}{\rho}\boldsymbol{\mathcal{L}}_{1*}\boldsymbol{B}_*\boldsymbol{N}_*\boldsymbol{A}_*^{-1}\boldsymbol{M}_*^{-1}\boldsymbol{M}^{-1}\boldsymbol{N}\,.$$

If we use the definitions (17a) and (17b), we obtain

$$\boldsymbol{\mathcal{R}}^{\infty} = \frac{1}{\rho}\boldsymbol{\mathcal{L}}_{1*}\boldsymbol{\tau}_*\boldsymbol{\Gamma}_*^{-1}\boldsymbol{M}^{-1}\boldsymbol{N}\,. \quad (69)$$

We now insert (69) into (68) and rearrange:

$$\begin{aligned} \mathbf{I} + z^{-1}\boldsymbol{\mathcal{F}}^{\infty} &= \boldsymbol{\mathcal{L}}_{1*}\left(\mathbf{I} + \frac{1}{\rho}\boldsymbol{\tau}_*\boldsymbol{\Gamma}_*^{-1}\boldsymbol{\Gamma}^{-1}\boldsymbol{\tau}\right) \\ &= \boldsymbol{\mathcal{L}}_{1*}\left(\mathbf{I} + \frac{1}{\rho}\tilde{\boldsymbol{\Gamma}}_*^{-1}\tilde{\boldsymbol{\tau}}_*\tilde{\boldsymbol{\tau}}\tilde{\boldsymbol{\Gamma}}^{-1}\right) \\ &= \boldsymbol{\mathcal{L}}_{1*}\tilde{\boldsymbol{\Gamma}}_*^{-1}\left(\tilde{\boldsymbol{\Gamma}}_*\tilde{\boldsymbol{\Gamma}} + \frac{1}{\rho}\tilde{\boldsymbol{\tau}}_*\tilde{\boldsymbol{\tau}}\right)\tilde{\boldsymbol{\Gamma}}^{-1} \\ &= \boldsymbol{\mathcal{L}}_{1*}\tilde{\boldsymbol{\Gamma}}_*^{-1}\boldsymbol{\beta}_*\mathbf{W}\boldsymbol{\beta}\tilde{\boldsymbol{\Gamma}}^{-1} \end{aligned}$$

where we in the second equality used the identity (18) and in the last equality the spectral factorization (29). We now collect polynomial matrices in z^{-1} on the left hand side and polynomial matrices in z on the right hand side:

$$(\mathbf{I} + z^{-1}\boldsymbol{\mathcal{F}}^{\infty})\tilde{\boldsymbol{\Gamma}}\boldsymbol{\beta}^{-1} = \boldsymbol{\mathcal{L}}_{1*}\tilde{\boldsymbol{\Gamma}}_*^{-1}\boldsymbol{\beta}_*\mathbf{W}\,. \quad (70)$$

The left hand side contains only powers of z^{-1} and the right hand side contains only powers of z. The only way for this equality to hold is to require that both sides equal a constant matrix. Since $\boldsymbol{\beta}$ is monic, we see that the constant term of the left hand side equals $\tilde{\Gamma}_0$, the constant term of $\tilde{\mathbf{\Gamma}}$. We must thus require

$$(\mathbf{I} + z^{-1}\boldsymbol{\mathcal{F}}^{\infty})\tilde{\mathbf{\Gamma}}\beta^{-1} = \tilde{\mathbf{\Gamma}}_0$$
$$\boldsymbol{\mathcal{L}}_{1*}\tilde{\mathbf{\Gamma}}_*^{-1}\boldsymbol{\beta}_*\mathbf{W} = \tilde{\mathbf{\Gamma}}_0 \ ,$$

or, equivalently

$$\boldsymbol{\mathcal{L}}_{1*} = \tilde{\mathbf{\Gamma}}_0\mathbf{W}^{-1}\boldsymbol{\beta}_*^{-1}\tilde{\mathbf{\Gamma}}_* \tag{71}$$
$$\boldsymbol{\mathcal{F}}^{\infty} = z(\tilde{\mathbf{\Gamma}}_0\beta\tilde{\mathbf{\Gamma}}^{-1} - \mathbf{I}) \ . \tag{72}$$

We can now insert (71) into (69) to arrive at our final expression for the asymptotic DFE filters:

$$\boldsymbol{\mathcal{R}}^{\infty} = \frac{1}{\rho}\tilde{\mathbf{\Gamma}}_0\mathbf{W}^{-1}\boldsymbol{\beta}_*^{-1}\tilde{\mathbf{\Gamma}}_*\boldsymbol{\tau}_*\mathbf{\Gamma}_*^{-1}\boldsymbol{M}^{-1}\boldsymbol{N} \tag{73a}$$
$$= \frac{1}{\rho}\tilde{\mathbf{\Gamma}}_0\mathbf{W}^{-1}\boldsymbol{\beta}_*^{-1}\tilde{\boldsymbol{\tau}}_*\boldsymbol{M}^{-1}\boldsymbol{N} \tag{73b}$$
$$\boldsymbol{\mathcal{F}}^{\infty} = z(\tilde{\mathbf{\Gamma}}_0\beta\tilde{\mathbf{\Gamma}}^{-1} - \mathbf{I}) \ . \tag{73c}$$

If we replace z and z^{-1} with q and q^{-1} respectively, (73a) coincides with (28a), (73b) coincides with (28b) and (73c) coincides with (28c).

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