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POLYNOMIAL METHODS IN OPTIMAL CONTROL AND FILTERING

Edited by K. J. HUNT

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Contents

Chapter 3

LQ Controller Design and Self-tuning Control

M. Sternad and A. Ahlén

3.1 Introduction

The present chapter contributes on some aspects of infinite–horizon LQG control for discrete time systems[1]. It takes up four themes, which complement mainly Chapter 2 and Chapter 5 of this volume:

1. A novel *variational technique*, for deriving polynomial equations for LQG controller design, is presented in Section 3.2. It is illustrated on MIMO feedforward design in Section 3.3. (Application of this tool to filtering problems is discussed in Chapter 5.)

2. Rejection of *nonstationary disturbances*, described by marginally stable and non–stabilisable models, is another topic. A control law discussed in Section 3.4 achieves this by utilising the internal model principle (generalised integration) combined with disturbance measurement feedforward.

3. The suitability of polynomial LQG design as a candidate for *self–tuning control* is highlighted, and an algorithm is presented in Sections 3.5.

4. We investigate some of the user choices in LQG control, which influence the *robustness* of both self–tuning and fixed control laws. In particular, the roles of feedforward control and of different choices of observer polynomials are discussed. See Section 3.6.

These contributions are four inter–related pieces of a larger puzzle, which we see as a challenge for research in the coming years. To explain our interest in the discussed issues, a view of the present state of the area is first outlined.

[1]The chapter describes work partially supported by the Swedish National Board for Technical Development (NUTEK), under grants 84-3680 and 87-01573. Parts of Sections 3.4 and 3.5, and most of Section 3.6, have previously appeared in the paper [53] in the International Journal of Control, published by Taylor and Francis Ltd.

3.1.1 The state of the art

Design of regulators for linear time–invariant discrete–time systems is an area where it is natural to use the polynomial approach, This was, historically, the domain where polynomial methods were first developed. Polynomial methods are now included in many standard textbooks on sampled data control. See e.g. [4].

Consider a simple scalar system with input $u(t)$, output $y(t)$ and output disturbance $n(t)$. It is expressed by polynomials in the backward shift operator q^{-1}:

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})}u(t-k) + n(t) \quad ; \quad k \geq 1 \ . \tag{3.1}$$

Control by a feedback regulator

$$R(q^{-1})u(t) = -S(q^{-1})y(t) + r(t) \tag{3.2}$$

results in the closed loop system

$$P(q^{-1})y(t) = B(q^{-1})r(t-k) + A(q^{-1})R(q^{-1})n(t) \tag{3.3}$$

where

$$P(q^{-1}) \triangleq A(q^{-1})R(q^{-1}) + q^{-k}B(q^{-1})S(q^{-1}) \ . \tag{3.4}$$

When a desired closed–loop system denominator $P(q^{-1})$ is given, equation (3.4) constitutes a linear polynomial equation, a *Diophantine equation.* This is one of the three basic types of equations which occur when regulators and filters are designed[2]. It is easily solved in the unknowns $R(q^{-1})$ and $S(q^{-1})$. Two nice features of the polynomial approach are evident in the feedback system above.

1. By manipulating numerator and denominator polynomials separately, the design equation (3.4) becomes linear in the unknowns, in spite of the fact that the regulator enters the closed–loop system in a nonlinear way. (This feature arises whenever a controlled object can be described in fractional form. See [31].)

2. The resulting system (3.3) is in input–output form. This is helpful for gaining immediate insight into some properties of the solution.

When $A(q^{-1})$ and $B(q^{-1})$ have no common factors, the closed–loop denominator $P(q^{-1})$ can, in principle, be chosen in an arbitrary way. For example, the choice $P(q^{-1}) = 1$ results in state deadbeat control, while $P(q^{-1}) = B(q^{-1})$ results in output deatbeat control, if the system is minimum phase. If the disturbance is described by the ARMA model

$$n(t) = \frac{C(q^{-1})}{A(q^{-1})}e(t)$$

[2]Polynomial spectral factorisations and the coprime factorisation of polynomial matrices in multivariable problems are the two other types.

where $e(t)$ is white noise, then the choice $P(q^{-1}) = B(q^{-1})C(q^{-1})$ results in minimum variance control for minimum phase systems. See e.g. [4].

With respect to polynomial methods, the level of knowledge around 1970 roughly corresponded to what has been outlined above, including significant multivariable generalisations [45], [57]. Since then, several workers have extended and generalised the polynomial approach to linear controller design, beginning with the minimum variance regulator for non–minimum phase systems of Peterka [42]. Central to this development was, of course, the work by Kučera [27], [28], [29].

During the last decade, considerable interest has been focused on linear quadratic methods, inspired by the solution presented by Kučera in [27] and, more recently, in [32]. In the scalar case above, this LQG solution implies pole placement in

$$P(q^{-1}) = \beta(q^{-1})C(q^{-1}) \qquad (3.5)$$

where $\beta(q^{-1})$ is a stable polynomial, obtained from a criterion–related spectral factorisation. For details, see Section 3.4 below.

Compared to the other pole placement rules mentioned above, LQG optimisation is superior as a framework for regulator design. It provides a direct tradeoff between input energy and disturbance rejection. Non–minimum phase dynamics will not cause instability because $\beta(z^{-1})$, as opposed to $B(z^{-1})$, is stable, under mild conditions. Extensions of this solution include feedforward links and the use of dynamic cost weighting. See e.g. [18], [20], [22], [23], [24] [33], [41], [47], [50], [52] and [53]. A recent result is the derivation of polynomial equations for solving the standard $\mathcal{H}_2$ control problem. That framework contains, in principle, all the previously studied LQ problem formulations as special cases. See [11], [25] and Chapter 2 of this volume. Polynomial methods have also found use for $\mathcal{H}_\infty$–optimisation (cf. Chapters 4 and 6), although that field is dominated by the state–space approach of Doyle and co–workers.

3.1.2 So where do we go from here?

Is the theory of polynomial methods for linear controller design complete? No, far from it. It is our strong belief that, in spite of the past progress, some of the most relevant questions for controller design have hardly been posed, and much less answered.

Work on the polynomial approach has so far mainly focused on problems of *systems theory*: deriving design equations, investigating conditions for problem solvability in an idealised sense, and (to some extent) developing numerical algorithms. What is needed is a complementary development, which rather belongs to the field of *control*: we now have some tools, but *how* should we use them? Are they even the right tools for solving the problems we are really interested in?

Take the pole placement equation (3.4) as an example. Algebraically, arbitrary pole placement might be allowed. In reality, a belief that arbitrary pole placement is

possible can be outright dangerous. Just consider the extremely high controller gain and sensitivity to model errors which can result from a state deadbeat design. Or think of minimum variance design, which, in addition to the above problems, frequently results in output oscillations between the sampling instants. (The use of LQG is better, but does not guarantee a good control behaviour.) If our goal is to convert the polynomial approach into a sensible design philosophy, much remains to be done. Some important design issues, which should be taken into account, have been discussed in an excellent way by Boyd and Barratt [10] and by Maciejowski [38]. A sample of challenging issues:

- Guidelines are needed on what kind of specifications it is futile to even ask for in a controller design. Today, many limitations on achievable performance are known, but we have no good ways of building them into the polynomial approach. For example, it is well known that non–minimum phase zeros impose upper bounds on the achievable performance of a closed–loop system. These bounds are of engineering nature, rather than algebraic; the roll–off rate could always be chosen so as not to obtain instability, but at the expense of disturbance amplification. The consideration of practical design constraints will often lead to rather different answers than consideration of purely algebraic constraints[3].

- Design involves tradeoffs; we often move in a Pareto–optimal subspace of the design parameter space. Control structures which introduce new degrees of freedom simplify the task for the designer. The study of their properties is of high practical relevance. One such control structure is the use of disturbance measurement feedforward; it is one of the main themes of this chapter.

- An LQG approach, in particular with frequency dependent weightings, might be a fruitful starting point in a quest for a sensible design philosophy. However, the role of design choices, such as dynamic weightings and observer poles, is still insufficiently understood. (A difficulty is that it is rather hard to "see through" Diophantine equations. The task does not exactly become easier when the solution involves a long list of coprime factorisations.) Some guidelines on these issues for SISO systems, based on our experience, are discussed in Section 3.6.

- Furthermore, the models on which control or filtering is based, must be obtained in some way. Here, the connection to system identification [36], [49], recursive identification [37] and adaptive methods is of importance. Such issues are discussed in Section 3.5.

- The question of robustness against modelling errors is perhaps the most difficult challenge. It should be noted that this problem does not disappear just because

[3] An example is the question if one single of two coupled Diophantine equations are required for each degree of freedom in LQG control. For the case of feedback in an adaptive setting, the use of two coupled equations should be preferred. Stable common factors will often appear in identified models, and the use of two equations avoids numerical problems for that reason. See Section 3.4. In feedforward control problems, on the other hand, no *realistic* problem formulation will include strictly unstable models of exogenous signals, which could cause solvability problems. There, the use of one single equation is sufficient. See Section 3.3 and 3.4.

adaptive algorithms are used, cf. Section 3.6. An LQG design can be arbitrarily sensitive, but the conditions when this actually occurs are not well understood[4]. A focus on *performance* robustness, as opposed to just stability robustness, would be desirable. Robust design has largely been ignored in the polynomial literature, with the exception of a recent paper by Grimble [21].

One of our convictions is that, to gain insight and design intuition, one needs to continue to study different special *explicit model* formulations. They often have much simpler and more transparent solutions that the most general problems. (See e.g. the filtering problems discussed in Chapter 5.) For this reason, one of our interests has been in simple derivation methods for obtaining solutions. The result of that work is presented in Section 3.2.

3.1.3 Remarks on the notation

The backward shift operator q^{-1} corresponds to z^{-1} in the frequency domain. Trace and transpose of matrices M will be denoted tr$M$ and M', respectively. $\mathcal{E}$ will denote expectation. For any polynomial matrix $P(z^{-1})$, $P_* = P(z)'$. The arguments will often be omitted, unless there is risk for confusion. A square polynomial matrix, of full normal rank, is called *stable* (or strictly Schur), if its determinant has all zeros in $|z| < 1$. Rational matrices $\mathcal{R}(z^{-1})$ are called stable if all their elements are transfer functions with poles in $|z| < 1$. If $P(q^{-1})$ is a square polynomial matrix, all elements of the rational matrix $P(q^{-1})^{-1}$ are *causal* if and only if the leading coefficient matrix of $P(q^{-1})$, denoted $P(0)$, is nonsingular. The *degree* of $P(q^{-1})$ is the highest degree of any of its polynomial elements.

3.2 Outline of a Variational Procedure for Deriving LQG Control Laws in Polynomial Form

A new way of deriving LQG controller design equations, which requires significantly fewer algebraic steps, compared to the traditional "completing the squares–method" [27], is presented here. Essentially, an old variational argument is utilised in a novel way. Orthogonality between signals and variations is evaluated in the frequency domain, to obtain polynomial equations which define the control law. The resulting equations are, of course, the same as obtained by a "completing the squares"–reasoning. Subsequent discussion of solvability of the equations and stability of the solution remains unaltered.

The method has been presented, and demonstrated on MIMO feedback design, in [54]. Application on control problems with marginally stable (and non-stabilizable) blocks is demonstrated here. That generalisation leaves the reasoning unchanged. It requires mainly a separate verification of finite cost. Examples of such verifications

[4]Much attention has instead been focused on the LQG/LTR modification. The properties, and the drawbacks, of this technique can be analysed by using a polynomial approach. See [9].

are found in Appendix A and B. The method is applied on a multivariable feedforward problem, and a restricted version of the general $\mathcal{H}_2$ problem, in Section 3.3. A scalar feedback and feedforward regulator is derived in Section 3.4. Application to filtering problems can be found in [1] and in Chapter 5 of this volume [2]. Please refer to that chapter for a comparative evaluation of different derivation techniques.

Consider the control of a linear discrete–time system, which is stochastic and time–invariant. Its inputs $u(t) \in R^m$ are to be calculated, based on linear combinations of measurable outputs $z(t) \in R^n$, so that the signals $y(t) \in R^p$ are controlled. Denote the regulator

$$u(t) = -\mathcal{R}(q^{-1})z(t) \tag{3.6}$$

where $\mathcal{R}(z^{-1})$ is a causal and rational $m|n$–matrix. It is to be designed so that the controlled system is stable and the infinite–horizon quadratic criterion

$$J = \mathcal{E}\{\mathrm{tr}Vy(t)(Vy(t))' + \mathrm{tr}Wu(t)(Wu(t))'\} \tag{3.7}$$

is minimised. Above, $V(q^{-1})$ and $W(q^{-1})$ represent polynomial weighting matrices, of dimensions $p|p$ and $m|m$, respectively. (The use of rational weighting matrices is straightforward, but requires additional coprime factorisations in the solution for multivariable problems.) Variational arguments will be used in order to minimize (3.7). For that purpose, introduce the *alternative regulator*

$$u(t) = -\mathcal{R}(q^{-1})z(t) + \nu(t) \tag{3.8}$$

where $\nu(t) \in R^m$ is a linear function of available data up to time t. The use of (3.8) results in the modified signals

$$\begin{aligned} y(t) &= y_o(t) + \delta y(t) \\ u(t) &= u_o(t) + \delta u(t) \end{aligned} \tag{3.9}$$

where $y_o(t)$ and $u_o(t)$ result from control by (3.6), while $\delta y(t)$ and $\delta u(t)$ are caused by the variation $\nu(t)$. The criterion can then be expressed as

$$J = J_o + 2J_1 + J_2 \tag{3.10}$$

where

$$\begin{aligned} J_o &= \mathcal{E}(\mathrm{tr}(Vy_o)(Vy_o)' + \mathrm{tr}(Wu_o)(Wu_o)') \\ J_1 &= \mathcal{E}(\mathrm{tr}(Vy_o)(V\delta y)' + \mathrm{tr}(Wu_o)(W\delta u)') \\ J_2 &= \mathcal{E}(\mathrm{tr}(V\delta y)(V\delta y)' + \mathrm{tr}(W\delta u)(W\delta u)') \ . \end{aligned} \tag{3.11}$$

The goal is now to select $\mathcal{R}$ so that J_1 vanishes. Then, the regulator (3.6) is optimal; no perturbation $\nu(t)$ could improve the performance, since J_o does not depend on $\nu(t)$ and since $J_2 \geq 0$.

So far, this reasoning is well known, see e.g. [56]. We now outline our novel contributions. Assume $Vy_o(t)$, $V\delta y(t)$, $Wu_o(t)$ and $W\delta u(t)$ to be stationary. (This has to be verified, in each particular problem.) Let ℓ be the dimension of the noise vector disturbing the system. By using Parsevals formula, $J_1 = 0$ can be expressed as

$$J_1 = \frac{1}{2\pi i} \oint_{|z|=1} \mathrm{tr}\mathcal{M}(z, z^{-1}) \frac{dz}{z} = 0 \tag{3.12}$$

where $\mathcal{M}(z, z^{-1})$ is a rational $\ell|\ell$ matrix. The relation (3.12) is fulfilled if *each element of* $\mathcal{M}(z, z^{-1})z^{-1}$ *is made analytic in* $|z| \leq 1$. Then, the scalar integrand $\mathrm{tr}\mathcal{M}(z, z^{-1})z^{-1}$ is also analytic in $|z| \leq 1$, so the integral vanishes. These ℓ^2 element-wise conditions determine $\mathcal{R}$[5]. By using right matrix fraction descriptions (MFD's), they can be satisfied collectively, as will be exemplified in the next section. This results in the polynomial matrix equations to be used in the design.

The choice of variational term $\nu(t)$ is arbitrary, except that the modified control law must remain causal, and the variational term must not destabilise the system. Representations of stationary signals can be written in innovations form[6], where a stationary innovations signal, $\epsilon(t)$, represents the most recent information at time t. Whatever could possibly be achieved by a variation of the regulator (3.6), could just as well be accomplished by adding a *feedforward from the innovations*, $\nu(t) = \mathcal{T}(q^{-1})\epsilon(t)$, to it. Such a control variation preserves stability, since feedback loops are unaffected. With $\mathcal{T}$, of dimension $m|\ell$, being stable and causal and $\epsilon(t)$ being stationary, such a variation will always be admissible.

When there are several separate measurable signal sources, for example reference generators and disturbances, a control structure with several degrees of freedom can be optimised. The trick is to add *several* variational terms in (3.8), one for each signal source. The result is a criterion with several cross terms. These should be set to zero separately. This technique will be exemplified in Section 3.4.

Note that the condition $J_1 = 0$ can be expressed as

$$\mathcal{E}((V\delta y)'(W\delta u)') \begin{pmatrix} Vy_o \\ Wu_o \end{pmatrix} = 0 \ .$$

The vector $((Vy_o)'(Wu_o)')'$ contains signals appearing in the criterion when the regulator (3.6) is used. It is required to be orthogonal to the vector of perturbations $((V\delta y)'(W\delta u)')$, caused by admissible variations of the control law.

The outlined procedure is a *constructive* derivation technique. Its initial steps are related to a proof by contradiction, first presented in [4]. In that approach, the

[5]This is a crucial insight. It would be hard to determine $\mathcal{R}$ from the scalar condition (3.12) directly.

[6]If signals are generated by unstable systems, we call it a generalised innovations form.

optimality of a filter or regulator, obtained by other means, is verified; it is demonstrated that (3.12) is fulfilled, because the integrand is analytic in $|z| \leq 1$.

3.3 Feedforward and Disturbance Decoupling

Assume a stable system to be represented by a model in right MFD form

$$y(t) = BA^{-1}u(t) + DE^{-1}w(t) \tag{3.13}$$

where $w(t) \in R^\ell$ is a vector of measurable disturbances. It is described by a model in right MFD form, with poles inside or on the unit circle (describing e.g. a sequence of step disturbances)

$$w(t) = GH^{-1}v(t) \ . \tag{3.14}$$

Here, $v(t) \in R^\ell$ is stationary white noise, with zero means and covariance matrix $\psi \geq 0$[7]. The polynomial matrices $B(q^{-1})$, $A(q^{-1})$, $D(q^{-1})$, $E(q^{-1})$, $G(q^{-1})$ and $H(q^{-1})$ have dimensions $p|m$, $m|m$, $p|\ell$, $\ell|\ell$, $\ell|\ell$ and $\ell|\ell$, respectively. Delays are included in the corresponding polynomials of $B(q^{-1})$ and $D(q^{-1})$. The pairs (B, A), (D, E) and (G, H) need not necessarily be right coprime. We assume $w(t)$, but not $y(t)$, to be measurable. Thus, $z(t) = w(t)$. The criterion (3.7) is to be minimised by using feedforward control. Assume the following:

A1. The polynomial matrices A, E and G are all stable and have nonsingular leading coefficient matrices. Thus, they have stable and causal inverses.

A2. There exists a stable $m|m$ right polynomial spectral factor $\beta(q^{-1})$, defined by

$$\beta_*\beta = B_*V_*VB + A_*W_*WA \tag{3.15}$$

with $\beta(0)$ nonsingular[8].

Stability of the system is a natural requirement in an open–loop problem. Stable invertibility of the disturbance description implies that it must have full rank; there should really exist ℓ independent noise sources. The conditions for the existence of a stable spectral factor β are mild, see the footnote below. To assure a finite criterion in the presence of nonstationary disturbances, two additional assumptions are introduced. Let $\{z_j = e^{i\omega_j}\}$ denote all zeros of H on the unit circle. They are

[7]The formulation can also be interpreted as a reference feedforward (servo) problem, or a combination of reference and disturbance feedforward. In a reference feedforward problem, $w(t)$ is the command signal and $-DE^{-1}w(t)$ is an ideal response model.

[8]Two conditions are, together, sufficient for A2 to be fulfilled.

1) The matrix $[B_*V_* \ \ A_*W_*]$ has full (normal) row rank m. This is a condition for the existence of a spectral factor, see [27]. It is fulfilled, for example, if all m inputs are penalized.

2) The greatest common left divisor of B_*V_* and A_*W_* has nonzero determinant on $|z| = 1$. This assures $\det \beta(z^{-1}) \neq 0$ on $|z| = 1$. The factor β is unique, up to a left orthogonal factor.

the frequencies where the rank of $H(e^{i\omega_j})$ drops below full rank.

A3. The criterion polynomial W in (3.7) is chosen such that $W(z_j^{-1}) = 0$.

A4. There exist right inverses $B^{\dagger}(z^{-1})$ to $B(z^{-1})$ at all $z = \{z_j\}$.

Note that for $m > 1$, condition A3 is stronger than just requiring W to have a zero at $\{z_j\}$. Assumptions A3 and A4 have interesting physical interpretations, discussed in Appendix A. In order to minimize (3.7), the perturbed feedforward regulator

$$u(t) = -\mathcal{R}w(t) + \nu(t) \tag{3.16}$$

with $\mathcal{R}$ of dimension $m|\ell$, is introduced. Since G is assumed stably invertible, $v(t)$ in (3.14) represents the innovations sequence in this problem. All admissible variations can then be expressed as $\nu(t) = \mathcal{T}v(t)$. The rational matrix $\mathcal{T}(q^{-1})$ must be causal and stable, but is otherwise arbitrary.

When the system is controlled by (3.16), outputs and inputs are given by (3.9), where

$$\begin{aligned} y_o(t) &= (DE^{-1} - BA^{-1}\mathcal{R})w(t) \quad ; \quad \delta y(t) = BA^{-1}\mathcal{T}v(t) \\ u_o(t) &= -\mathcal{R}w(t) \quad ; \quad \delta u(t) = \mathcal{T}v(t) \ . \end{aligned} \tag{3.17}$$

The signals are stationary for any stable $\mathcal{R}$, since E^{-1}, A^{-1} and $\mathcal{T}$, are assumed stable. The use of (3.17) in (3.11) gives the cross–term

$$\begin{aligned} J_1 \ = \ & \frac{1}{2\pi i}\oint_{|z|=1} \mathrm{tr}[V(DE^{-1} - BA^{-1}\mathcal{R})GH^{-1}\psi\mathcal{T}_*A_*^{-1}B_*V_*]\frac{dz}{z} \\ & -\frac{1}{2\pi i}\oint_{|z|=1} \mathrm{tr}[W\mathcal{R}GH^{-1}\psi\mathcal{T}_*W_*]\frac{dz}{z} \ . \end{aligned} \tag{3.18}$$

By using $\mathrm{tr}\mathcal{A}\mathcal{B}_* = \mathrm{tr}\mathcal{B}_*\mathcal{A}$, (with $\mathcal{B} = A_*^{-1}B_*V_*$ and W_*, respectively), the matrices in both terms get equal dimension $\ell|\ell$[9]. The use of the spectral factorisation (3.15) to simplify the integrand then gives

$$J_1 = \frac{1}{2\pi i}\oint_{|z|=1} \mathrm{tr}[A_*^{-1}(B_*V_*VDE^{-1}G - \beta_*\beta A^{-1}\mathcal{R}G)H^{-1}\psi\mathcal{T}_*]\frac{dz}{z} \ . \tag{3.19}$$

When (3.19) is set to zero, it corresponds to (3.12). Note that E^{-1}, A^{-1} and H^{-1} have elements with poles only in $|z| \le 1$ and $1/z$ contributes a pole at $z = 0$. Poles at $z = 0$ can be caused by V, D, G, β, since they contain polynomials in z^{-1}. They must *all* be eliminated. For that purpose, introduce a right coprime factorisation

[9] This trace rotation step is not required in the (otherwise very similar) reasoning for optimising filters in [1] of Chapter 5 [2].

$$G_2E_2^{-1} = E^{-1}G \tag{3.20}$$

with G_2 and E_2 both of dimension $\ell|\ell$. Since E^{-1} is stable and causal, so is E_2^{-1}. The position of $\mathcal{R}$ in (3.19) enables direct cancellation of βA^{-1}, if $A\beta^{-1}$ is a left factor of $\mathcal{R}$. With $E_2^{-1}G^{-1}$ as a right factor of $\mathcal{R}$, G is also eliminated, while E_2^{-1} has to be factored out to the right, to be cancelled later. The regulator becomes

$$\mathcal{R} = A\beta^{-1}QE_2^{-1}G^{-1} \tag{3.21}$$

which is stable and causal, since β^{-1}, E_2^{-1} and G^{-1} are stable and causal. The $m|\ell$ polynomial matrix $Q(z^{-1})$ is not yet specified. Thus,

$$J_1 = \frac{1}{2\pi i}\oint_{|z|=1} \mathrm{tr}[A_*^{-1}(B_*V_*VDG_2 - \beta_*Q)E_2^{-1}H^{-1}\psi\mathcal{T}_*]\frac{dz}{z} \ . \tag{3.22}$$

The elements of $A_*^{-1}(z)$ and $\mathcal{T}_*(z)$ have poles strictly outside $|z| = 1$, since A and $\mathcal{T}$ are stable. Thus, all elements of the integrand become analytic inside $|z| = 1$ if there exists a polynomial matrix $L_*(z)$, of dimension $m|\ell$, such that

$$(B_*V_*VDG_2 - \beta_*Q)E_2^{-1}H^{-1}\frac{1}{z} = L_* \tag{3.23}$$

or

$$B_*V_*VDG_2 = \beta_*Q + L_*zHE_2 \ . \tag{3.24}$$

This is a bilateral Diophantine equation in $Q(z^{-1})$ and $L_*(z)$. Thus, the regulator can be obtained by solving (3.15) for β, computing G_2 and E_2 from (3.20), solving (3.24) for Q and L_*, and using the control law $u(t) = -A\beta^{-1}QE_2^{-1}G^{-1}w(t)$.

The reasoning, from (3.17) up to equation (3.24), constitutes a compact derivation of the optimal control law. The finiteness of the minimal cost is verified in Appendix A, under Assumptions A3 and A4.

The single Diophantine equation (3.24) determines the regulator uniquely[10]. Since $\det \beta_*(z)$ has zeros strictly outside $|z| = 1$, while $\det H(z^{-1})E_2(z^{-1})$ has zeros only in $|z| \le 1$, the invariant polynomials of β_* are coprime with all those of HE_2. Thus, a solution $(Q^o(z^{-1}), L_*^o(z))$ to (3.24) always exists. See Lemma 1 of [44]. All solutions can be expressed as

$$(Q, L_*) = (Q^o - XzHE_2, L_*^o + \beta_*X)$$

[10] This holds in general for *open-loop* control and estimation problems, if the involved systems are stable or marginally stable. If $\det H(z^{-1})$ had zeros in $|z| > 1$, two coupled Diophantine equations would sometimes be required to determine $\mathcal{R}$, and formally assure a finite criterion value J_0. (See subsection 5.3.9 of Chapter 5 for a discussion.) However, such control laws, designed to cancel exponentially increasing disturbances, are of no practical interest. Actuator limitations and sensitivity problems would defeat any such ambitions.

where the polynomial matrix X is arbitrary [27]. However, causality requires $Q(z^{-1})$ to have only nonpositive powers of z as arguments, while optimality requires $L_*(z)$ to have no negative powers of z as arguments. (If it had, it would contribute zeros at the origin in (3.22).) Thus, $X = 0$ is the only choice, so the solution to (3.24) is unique.

In SISO problems, with $V = 1$, $W = \rho\tilde{\Delta}(q^{-1})$, $G = G_2$ and $A = E = E_2$, (3.21) becomes $\mathcal{R} = Q/\beta G$, while (3.24) reduces to equation (3.12) in [52].

If the weights V and W in the criterion are rational, two additional coprime factorisations are required. It is shown in [7] that this more general problem is *dual to the generalised deconvolution problem* discussed in Section 5.4. It is very simple to demonstrate this duality. Take the block diagram for one of the problems. Reverse all arrows, interchange summation points and node points and transpose all rational matrices. Then, the block diagram for the other problem has been obtained.

The disturbance measurement is often influenced by the control signal. We then have a more general disturbance rejection problem, sometimes called *disturbance decoupling*. If the transfer function between $u(t)$ and the measurement is stable, it can be *subtracted internally*, inside the regulator. Assume the measurement to be given by

$$z(t) = \mathcal{H}_{zu}u(t) + \mathcal{H}_{zw}w(t) \tag{3.25}$$

with $\mathcal{H}_{zu}$ *stable*, rational and causal, $\mathcal{H}_{zw}$ *stably invertible* and $w(t)$ given by (3.14). The use of the control law

$$u(t) = -\mathcal{R}\mathcal{H}_{zw}^{-1}[z(t) - \mathcal{H}_{zu}u(t)] \tag{3.26}$$

reduces this problem to the already solved feedforward control problem, with the optimal $\mathcal{R}$ given by (3.21). This solution is somewhat related to Internal model control (IMC) [40]. A SISO version of it has been discussed in [50] and [51]. The accuracy of the model of $\mathcal{H}_{zu}$ must be good at frequencies where $\mathcal{R}\mathcal{H}_{zw}^{-1}$ has high gain. Otherwise, model imperfections could cause instability, as in all control laws.

This problem can be interpreted as a somewhat specialised standard $\mathcal{H}_2$ problem, represented by

$$\begin{pmatrix} \begin{bmatrix} Vy(t) \\ Wu(t) \end{bmatrix} \\ z(t) \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} VDE^{-1}GH^{-1} \\ 0 \end{bmatrix} & \begin{bmatrix} VBA^{-1} \\ W \end{bmatrix} \\ \mathcal{H}_{zw}GH^{-1} & \mathcal{H}_{zu} \end{pmatrix} \begin{pmatrix} v(t) \\ \\ u(t) \end{pmatrix}$$

with $\| (Vy)' \; (Wu)' \|_2^2$ being minimised and $z(t)$ measured. We assume $E^{-1}, A^{-1}, \mathcal{H}_{zw}$ and $\mathcal{H}_{zu}$ to be stable, while H^{-1} is stable or marginally stable. While not completely general, the formulation above, with a stable system, includes a large class of practically relevant control problems. The above solution, with *one* spectral factorisation, *one* coprime factorisation and *one* Diophantine equation, is evidently much simpler than the general polynomial $\mathcal{H}_2$ solutions known to date.

3.4 A Scalar Feedback–feedforward LQG Design

The polynomial equations approach to the design of combined feedback–feedforward regulators has received considerable interest. See [43], [47], [52] and [20]. The earliest result, restricted to random–walk disturbance models, was due to Peterka [43].

We will here use the methodology of Section 3.2 to derive a combined feedback–feedforward control law for a scalar plant. This plant may be affected by non-stationary measurable and unmeasurable disturbances, described by systems with poles on the unit circle. The adaptive controller of Section 3.5 will be based on this control law.

3.4.1 The control problem

Let the plant be described by the following linear discrete–time scalar model

$$Ay(t) = Bu(t-k) + Dw(t-d) + Cn(t) \tag{3.27}$$

where the output $y(t)$, input $u(t)$, measurable disturbance $w(t)$ and unmeasurable disturbance $n(t)$ are all scalar signals. All polynomials except $B(q^{-1})$ and $D(q^{-1})$, of degree na, nb etc, are monic. The delays are $k > 0$ and $d \geq 0$.

The disturbances $w(t)$ and $n(t)$ are asumed to be described by

$$w(t) = \frac{G}{H}v(t) = \frac{G}{H_S H_U}v(t) \;\; ; \;\; n(t) = \frac{1}{F}e(t) \;\; . \tag{3.28}$$

We assume $v(t)$ and $e(t)$ to be stationary, mutually uncorrelated and zero mean. They are stationary white noises or random spike sequences, with variance λ_v and λ_e, respectively. While $C(z^{-1}), G(z^{-1})$ and $H_S(z^{-1})$ are assumed to be stable, $H_U(z^{-1})$ and $F(z^{-1})$ have all their zeros on the unit circle. The disturbance models thus include

1. *Stationary stochastic disturbances.* (F or $H_U = 1$.)
2. *Drifting stochastic disturbances.* If $w(t)$ e.g. has stationary increments, it is modelled by $H_U = 1 - q^{-1}$ and a white noise $v(t)$.
3. *Shape–deterministic* or piecewise deterministic signals, such as random step sequences, ramp sequences or sinusoids which occasionally change magnitude or phase. A stationary random spike sequence, such as a Bernoulli–Gaussian sequence [11], is then a reasonable model for $v(t)$ or $e(t)$. (Purely deterministic disturbances can also be included in the formulation. See [53]. For an alternative problem formulation, see [35].)

[11] A Bernoulli–Gaussian sequence is given by $v(t) = r(t)s(t)$ where $s(t)$ is a Bernoulli sequence such that $s(t) = 1$ w. p. λ and $s(t) = 0$ w. p. $1 - \lambda$. $r(t)$ is a zero mean Gaussian sequence with variance σ^2 independent of t, cf. [39]. It is then straightforward to show that $v(t)$ is a stationary white sequence with zero mean and variance $\lambda_v = \sigma^2\lambda$.

Assume, for now, all polynomials to be known. The goal is to minimize an infinite horizon criterion

$$J = \mathcal{E}(Vy(t))^2 + \rho\mathcal{E}(WFu(t))^2 \quad . \tag{3.29}$$

The input penalty $\rho \geq 0$ and the polynomials $V(q^{-1})$ and $W(q^{-1})$ are chosen by the designer. Note that the choice of input filters is not completely free: the factor $F(q^{-1})$ must be present whenever $n(t)$ is described by a marginally stable model. If $n(t)$ e. g. is a drifting signal, a drifting input $u(t)$ will be needed. To keep the criterion finite, the input must then be filtered by $F(q^{-1}) = 1 - q^{-1}$ in (3.29).

In this problem, the measurement is $z(t) = (y(t)\ w(t))'$. It will be shown that the optimal linear regulator structure, with feedback and feedforward, is given by

$$RFu(t) = -\frac{Q}{P}w(t) - Sy(t) \quad . \tag{3.30}$$

See Figure 3.1. The polynomial $P(q^{-1})$ is required to be stable. Note that the filter $1/R(q^{-1})F(q^{-1})$ is present in both the feedback and feedforward signal paths. The filtering by $1/F(q^{-1})$ is consistent with the internal model principle [15]. When $F(q^{-1}) = 1 - q^{-1}$, we have an integrating regulator with a feedforward term.

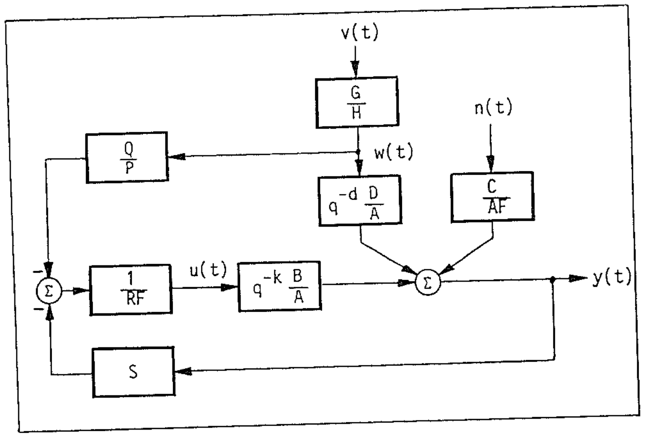

Figure 3.1: The system and regulator structure.

The regulator design will consist of a simple two-step procedure. First, the feedback $\{R, S\}$ is optimised with respect to the unmeasurable disturbance $n(t)$. Then, the feedforward filter $\{P, Q\}$ is calculated so that the measurable disturbance $w(t)$ is rejected in an optimal way. This separability is possible since $w(t)$ and $n(t)$ are assumed uncorrelated. The feedback design equations are well-known [27], but are slightly generalised here to cover $F \neq 1$.

3.4.2 The design equations

Introduce the polynomial spectral factorisation

$$r\beta\beta_* = BVV_*B_* + \rho AFWW_*F_*A_* \tag{3.31}$$

where r is a positive scalar and $\beta(z^{-1})$ is a stable monic polynomial with degree $n\beta$. When $\rho > 0$, stability of β is assured if BV and AFW have no common factors with zeros on $|z| = 1$. If $\rho = 0$, BV should have no zeros on $|z| = 1$.

The following assumptions are sufficient for the existence of a unique stabilizing solution to the optimisation problem described above.

B1. The polynomials β, C and G are stable.

B2. The polynomials AF and B have no unstable common factors.

B3. H_U is a factor of WFD.

Theorem 3.1

Under the conditions B1–B3 above, the controlled system (3.27),(3.30) attains the global minimum of (3.29), under the constraint of stability, if $\{R, S, P, Q\}$ are calculated as follows.

Let $R(q^{-1}), S(q^{-1})$ and $X_*(q)$ be the unique solution of the coupled linear polynomial equations

$$r\beta_*R - q^{-k+1}BX_* \;=\; \rho WW_*F_*A_*C \tag{3.32}$$

$$r\beta_*S + qAFX_* \;=\; q^kCVV_*B_* \quad . \tag{3.33}$$

Let

$$P = G$$

and let $Q(q^{-1})$, together with $L_*(q)$, be the unique solution of

$$q^{-d+1}DFGX_* = r\beta_*Q + qCHL_* \quad . \tag{3.34}$$

□

3.4.3 Proof of Theorem 3.1

We add a variational term $\nu(t)$ to a linear regulator with general structure

$$u(t) = -\frac{S_1}{R_1}y(t) - \frac{Q_1}{P_1}w(t) + \nu(t) \tag{3.35}$$

As mentioned in Section 3.2, the variational term is set to a sum of potential feedforward contributions from the two innovation signals in the problem structure

$$\nu(t) = \mathcal{T}_1 v(t) + \mathcal{T}_2 e(t) \quad . \tag{3.36}$$

The use of (3.35) and (3.36) on the system (3.27), (3.28) results in the following expressions for the nominal closed-loop signals and their perturbations (3.9)

$$\begin{aligned} y_o(t) &= \frac{NG}{P_1\alpha H}v(t) + \frac{R_1C}{\alpha F}e(t) \;;\quad \delta y(t) = q^{-k}\frac{BR_1}{\alpha}(\mathcal{T}_1 v(t) + \mathcal{T}_2 e(t)) \\ u_o(t) &= -\frac{UG}{P_1\alpha H}v(t) - \frac{S_1C}{\alpha F}e(t) \;;\quad \delta u(t) = \frac{AR_1}{\alpha}(\mathcal{T}_1 v(t) + \mathcal{T}_2 e(t)) \end{aligned} \tag{3.37}$$

where

$$N \triangleq (q^{-d}DP_1 - q^{-k}BQ_1)R_1 \;;\quad \alpha \triangleq AR_1 + q^{-k}BS_1 \;;\quad U \triangleq AQ_1R_1 + q^{-d}DS_1P_1 \; .$$

The polynomial α must be stable, and will be required to be monic. Using (3.37) in (3.11), the condition for optimality can be expressed as

$$J_1 = \mathcal{E}(Vy_o)(V\delta y) + \rho\mathcal{E}(WFu_o)(WF\delta u) =$$

$$= \mathcal{E}\left\{V\left(\frac{NG}{P_1\alpha H}v(t) + \frac{R_1C}{\alpha F}e(t)\right) Vq^{-k}\frac{BR_1}{\alpha}(\mathcal{T}_1 v(t) + \mathcal{T}_2 e(t))\right\}$$

$$+ \rho\mathcal{E}\left\{WF\left(-\frac{UG}{P_1\alpha H} - \frac{S_1C}{\alpha F}e(t)\right) WF\frac{AR_1}{\alpha}(\mathcal{T}_1 v(t) + \mathcal{T}_2 e(t))\right\} = 0$$

In order to obtain a two degree of freedom controller, we set the contributions from the two variational terms $\mathcal{T}_2 v(t)$ and $\mathcal{T}_1 e(t)$ to zero *separately*, by tuning feedback and feedforward filters, respectively [12]. By using the assumption that $v(t)$ and $e(t)$ are uncorrelated, and by using the spectral factorisation (3.31), we obtain

Term 1:

$$\mathcal{E}\left\{\left(\frac{VNG}{P_1\alpha H}v(t)\right) Vq^{-k}\frac{BR_1}{\alpha}\mathcal{T}_1 v(t)\right\} - \rho\mathcal{E}\left\{\left(\frac{WFUG}{P_1\alpha H}v(t)\right) W\frac{FAR_1}{\alpha}\mathcal{T}_1 v(t)\right\} =$$

$$\frac{\lambda_v}{2\pi i}\oint \frac{G[V(z^{-d}DP_1 - z^{-k}BQ_1)R_1V_*z^kB_* - \rho WF(AQ_1R_1 + z^{-d}DS_1P_1)W_*F_*A_*]R_{1*}}{P_1\alpha H\alpha_*} .\mathcal{T}_{1*}\frac{dz}{z} =$$

$$\frac{\lambda_v}{2\pi i}\oint \frac{G[z^{-d+k}VV_*DR_1P_1B_* - r\beta\beta_*Q_1R_1 - \rho WW_*FF_*z^{-d}DS_1P_1A_*)]R_{1*}}{P_1\alpha H\alpha_*}\mathcal{T}_{1*}\frac{dz}{z} = 0 \tag{3.38}$$

[12] Note that the part affected by $e(t)$ only (term 2 below) can vanish only by choosing the feedback properly.

Term 2:

$$\mathcal{E}\left\{\left(\frac{VR_1C}{\alpha F}e(t)\right) Vq^{-k}\frac{BR_1}{\alpha}\mathcal{T}_2 e(t)\right\} - \rho\mathcal{E}\left\{\left(\frac{WFS_1C}{\alpha F}e(t)\right) WF\frac{AR_1}{\alpha}\mathcal{T}_2 e(t)\right\} =$$

$$\frac{\lambda_e}{2\pi i}\oint \frac{(R_1CVV_*z^kB_* - \rho WFS_1CW_*F_*A_*)R_{1*}}{\alpha\alpha_*F}\mathcal{T}_{2*}\frac{dz}{z} = 0 \; . \tag{3.39}$$

In order to find the *optimal feedback*, we shall set term 2, which depends on $e(t)$, to zero. All poles of the integrand of (3.39) within $|z| \le 1$ are cancelled by zeros if there exists a polynomial $X_*(z)$, such that

$$R_1CVV_*z^kB_* - \rho WW_*F_*A_*CFS_1 = X_*z\alpha F = X_*z(AR_1 + z^{-k}BS_1)F \tag{3.40}$$

or

$$R_1(z^kCVV_*B_* - zX_*AF) = (\rho WW_*F_*A_*C + z^{-k+1}BX_*)FS_1 \; .$$

If R_1 and S_1 are assumed coprime, there must exist a polynomial $K_*(z, z^{-1})$, such that

$$F(\rho WW_*F_*A_*C + z^{-k+1}BX_*) = K_*R_1 \tag{3.41}$$

$$z^kCVV_*B_* - zAFX_* = K_*S_1 \; . \tag{3.42}$$

Multiply (3.41) by A and (3.42) by $z^{-k}B$ and add them. This gives

$$Cr\beta\beta_* = K_*\alpha \; .$$

Thus, we obtain $\alpha = \beta C$ (stable) and $K_* = r\beta_*$, since α is required to be stable and monic. Note that F in (3.41) must be a factor of $K_*R_1 = r\beta_*R_1$. Since β_* has no zeros on the unit circle, F must be a factor of R_1. Thus, set

$$R_1 = RF \;;\quad S = S_1 \; . \tag{3.43}$$

Use (3.43) and $K_* = r\beta_*$ in (3.41), (3.42) and substitute q for z. After F has been cancelled in (3.41), they are seen to be identical to the feedback design equations (3.32), (3.33). This completes the feedback derivation.

In order to find the *optimal feedforward* polynomials, we set the first term (3.38), which depends on $v(t)$, to zero. The integrand is analytic in $|z| \le 1$, if there exists a polynomial $L_*(z)$, such that

$$z^{-d+k}GVV_*DR_1P_1B_* - Gr\beta\beta_*Q_1R_1 - \rho GWW_*FF_*z^{-d}DS_1P_1A_* = zL_*P_1\alpha H \; .$$

A rearrangement of the terms gives, with $R_1 = RF$ and $S_1 = S$,

$$P_1(z^{-d+k}GVV_*DRFB_* - \rho GWW_*FF_*z^{-d}DA_*S - zL_*\alpha H) = Gr\beta\beta_*Q_1RF \tag{3.44}$$

The feedforward denominator P_1 must be a factor of the right-hand side. To preserve stability, factors of P_1 which are not included in the feedback loop i.e. as factors of

$R_1 = RF$, must be stable. With Q_1 being a filter polynomial and β_* unstable, we may set

$$P_1 = RFG\beta \tag{3.45}$$

By cancelling P_1, (3.44) reduces to a Diophantine equation

$$(VV_*RB_* - \rho WW_*F_*A_*Sz^{-k})z^{-d+k}DGF = r\beta_*Q_1 + zL_*\alpha H \ . \tag{3.46}$$

This is the feedforward desing equation, to be solved for $Q_1(z^{-1})$ and $L_*(z)$. The equation depends on the feedback polynomials R and S. It can be used for optimising a feedforward filter which works in conjunction with an *arbitrary* feedback regulator, e.g. PID-type. (For $V = 1$ and $F = 1$, it corresponds to (3.7) in [52].)

When used in conjunction with an LQG feedback, which results in pole placement in $\alpha = \beta C$, the equation can be further simplified. Note that by using $R_1 = RF, S_1 = S$ and $\alpha = \beta C$ in (3.40), that equation can be written

$$(VV_*RB_* - \rho WW_*F_*A_*Sz^{-k})z^kCF = X_*z\beta CF \ .$$

By cancelling CF above, we obtain a new expression for the left-hand side of (3.46). That equation now becomes

$$z^{-d+1}DGFX_*\beta = r\beta_*Q_1 + zL_*\beta CH \ . \tag{3.47}$$

Since β is a factor of two terms, it must be a factor of Q_1 as well, i.e. $Q_1 = Q\beta$. Thus,

$$\frac{Q_1}{P_1} = \frac{Q\beta}{RFG\beta} = \frac{Q}{RFG} \ .$$

Therefore, $P = G$, (3.47) equals (3.34) and the controller structure (3.30) is verified. By using (3.43), (3.45), $Q_1 = \beta Q$ and $\alpha = \beta C$ in (3.37), the closed loop is found to be stable, except for the poles belonging to H_u and F. They belong to the external, non-stabilisable, disturbance-generating systems. In Appendix B, it is verified that the minimal cost is finite, under condition B3. □

3.4.4 Remarks and interpretations

Degrees of the feedback equations.

The variables in (3.32) and (3.33) have degrees

$$\begin{aligned} nx &= n\beta + k - 1 \\ ns &= \max\{nf + na - 1, nc - k\} \\ nr &= \begin{cases} \max\{nb + k - 1, nc + nw\} & \text{if } \rho \neq 0 \\ nb + k - 1 & \text{if } \rho = 0 \end{cases} \end{aligned} \tag{3.48}$$

For a discussion of the unique solvability of (3.32) and (3.33), see [27].

Interpretation of the conditions B1–B3.

In the proof above, optimal feedback was found to imply pole placement in βC:

$$AFR + q^{-k}BS = \beta C \ . \tag{3.49}$$

In addition, the feedforward filter introduces poles in the zero locations of $P = G$. This explains condition B1. If AF and B have no common factors, an optimal feedback can be calculated from the implied pole placement equation (3.49). With common factors, this will not be possible [30]. (We cannot cancel common factors and utilise e.g. the minimum degree solution. That solution would not satisfy the original equations.) The equations (3.32) and (3.33) do, however, give the correct solution, as long as the common factors of AF and B are stable.

Condition B2, the absence of unstable common factors of AF and B, is the condition for solvability of (3.32) and (3.33). It also corresponds to stabilizability and detectability of an equivalent system, with $Fu(t)$ as input, obtained by multiplication of (3.27) by F:

$$(AF)y(t) = q^{-k}B(Fu(t)) + q^{-d}(DF)w(t) + Ce(t) \ .$$

The condition B3, that H_U divides WFD, guarantees that the minimal cost is finite. See Appendix B. It has a natural explanation. To be more specific, let $w(t)$, with $H_U(q^{-1}) = 1 - q^{-1} \triangleq \Delta(q^{-1})$, be a drifting stochastic disturbance, i.e. $w(t) = w(t-1) + (G/H_S)v(t)$. Let $V(q^{-1}) = 1$. The controller must then eliminate drifts from the signals $y(t)$ and $WFu(t)$, which appear in the criterion. Otherwise, the criterion would be infinite.

- If $H_U = \Delta$ is a factor of D, the non-stationary mode of $w(t)$ is blocked before it reaches the output. Consequently, it does not result in a drifting $y(t)$. Since Δ is a factor of D and H in (3.34), it will also be a factor of Q. Thus, the feedforward control signal $(Q/P)w(t)$ remains stationary.
- If $H_U = \Delta$ is a factor of F, we have an integrating feedback regulator. It eliminates drifts in $y(t)$ caused by drifts in $w(t)$ (or in $n(t)$), regardless of the presence of any feedforward filter. The signal $Fu(t)$ appearing in the criterion (but not $u(t)$ itself), remains stationary.
- If $H_U = \Delta$ is not a factor of DF, the responsibility for eliminating drifts in $y(t)$ is placed on the feedforward filter. To accomplish this task, the filter generates a drifting control signal $(Q/P)w(t)$. Consequently, Δ must be a factor of W, so that $Wu(t)$ gives a finite contribution to the criterion.

When the disturbance $w(t)$ is described by a model with poles on the unit circle, such effects on $y(t)$ (e.g. static errors, drifts or undamped sinusoids) can be controlled either by the feedback or by the feedforward action. Use of the feedback, when H_U

divides F, results in a "robust" disturbance rejection. It is robust in the sense that the criterion is finite if modelling errors in A, B, C or D are present, as long as the closed–loop system is stable. However, F itself must be known exactly. When the feedforward action is used, the magnitude of static errors, drifts or sinusoids can be reduced, but can not be eliminated completely in practice. Modelling errors will cause imperfect cancellation.

The feedforward control.

Note that the solution of only one additional Diophantine equation, namely (3.34), is needed for optimising a feedforward filter. Since β (stable) and $C_* H_*$ (unstable) cannot have common factors, (3.34) is always solvable. The degrees of $Q(z^{-1})$ and $L_*(z)$ are defined uniquely by the requirement that they should cover the maximal occuring powers of z^{-1} and z, respectively, in (3.34):

$$\begin{aligned} nQ &= \max\{nd + nf + ng + d, nc + nh\} - 1 \\ nL &= \max\{0, k - d\} + n\beta - 1 \ . \end{aligned} \tag{3.50}$$

(See also Theorem 5.3.1 in Chapter 5 of this volume, or Lemma 1 in [1].)

The addition of a feedforward link simplifies the control task remaining for the feedback. (See also Section 3.6.) The delay d affects the achievable control quality significantly. It can be shown that application of feedforward can always improve the control performance when $d > 0$, compared to feedback from $y(t)$ only. The improvement is a nondecreasing function of d. It is advantageous to place the $w(t)$–sensor so that the disturbance is captured as early as possible, i. e. d is large.

Complete elimination of the measurable disturbance can be achieved if and only if $d \geq k$, and all unstable factors of $B(q^{-1})$ are factors also of $D(q^{-1})$. Any part of the system with non–minimum phase dynamics is then located beyond the point where the disturbance meets the control action. See [51], Section 3.3.

It is straightforward to generalise the solution to multiple measurable disturbances. One additional scalar Diophantine equation (3.34) is then obtained for each disturbance. The general multivariable result corresponding to Theorem 3.1 is presented in the report version of [54]. An alternative solution is derived in [24].

Numerical aspects.

The regulator should be realised minimally, as a single dynamical system having two inputs and one output. A common special case is when the measurable disturbance is drifting or of random step type, and an integrating regulator is used. Then, $H_U = F = \Delta$. Since Δ becomes a factor of both the left-hand side and rightmost term in (3.34), it must also be a factor of Q. With $Q = Q_2\Delta$, equation (3.34) is reduced to

$$z^{-d+1} DGX_* = r\beta_* Q_2 + zCH_S L_* \tag{3.51}$$

In this case, the controller (3.30) must be modified slightly. It can be implemented in differential form, using an explicit differentiation of the measurable disturbance:

$$\begin{aligned} R(\Delta u(t)) &= -\frac{Q_2}{G}(\Delta w(t)) - Sy(t) \\ u(t) &= u(t-1) + \Delta u(t) \ . \end{aligned} \tag{3.52}$$

Alternatively, one can use a structure with the feedforward filter separated from the integration:

$$Ru(t) = -\frac{Q_2}{G} w(t) - \frac{S}{\Delta} y(t) \ . \tag{3.53}$$

If (3.34) were used, small numerical errors and finite word–length effects would cause $Q \neq Q_2\Delta$. This could lead to large errors in the low-frequency gain of the feedforward filter $-Q/R\Delta P$ in (3.30). Design from (3.51), with $nQ_2 = nQ - 1$, and realisation according to (3.52) or (3.53), avoids such problems. Equation (3.34) must, however, be used in the general case, when $H_U \neq F$.

Reliable algorithms for polynomial spectral factorisation can be found in [27] and [26]. They are iterative, requiring typically 3-10 iterations, when starting from $\beta = 1$. In adaptive control, β from the previous controller calculation can be used as initial value. Then, normally only 1–2 iterations per updating are required.

The coupled equations (3.32),(3.33) represent an over–determined set of simultaneous equations in the coefficients of R, S and X. The system will, however, have a unique solution [27]. (Some equations are linear combinations of the others.) This (exact) solution can be found by computing the least–squares solution to the overdetermined system. Equation (3.34), with polynomial degrees (3.50), corresponds to a square system of linear equations, with full rank.

3.4.5 A numerical example

Example 3.1.

Consider the system

$$(1 - 0.9q^{-1})y(t) = (0.1 + 0.08q^{-1})u(t-2) + (0.2 + 0.4q^{-1})w(t-2) + e(t)$$

with $w(t) = w(t-1) + v(t)$ i.e. $H = 1 - q^{-1}$. Unit step disturbances $w(t)$ cause output deviations with amplitude ≈ 6 in this system. We design a regulator (3.30), such that the criterion (3.29), with $V = 1$ and with differential input penalty $\rho = 0.1, W = 1 - q^{-1}$, is minimized.

The spectral factorisation (3.31) becomes

$$r(1+\beta_1 q^{-1}+\beta_2 q^{-2})(1+\beta_1 q+\beta_2 q^2) = (0.1+0.08q^{-1})(0.1+0.08q) + 0.1(1-0.9q^{-1})(1-q^{-1})(1-q)(1-0.9q)$$

with solution $r = 0.2670$, $\beta_1 = -0.9887$ and $\beta_2 = 0.3370$.

The feedback part of the regulator is calculated from (3.32) and (3.33)

$$\begin{aligned} r(1+\beta_1 q+\beta_2 q^2)R(q^{-1}) - q^{-1}(0.1+0.08q^{-1})X_*(q) &= 0.1(1-q^{-1})(1-q)(1-0.9q) \\ r(1+\beta_1 q+\beta_2 q^2)S(q^{-1}) + q(1-0.9q^{-1})X_*(q) &= q^2(0.1+0.08q) \ . \end{aligned}$$

The variables have degree $nx = 3, ns = 0$ and $nr = 2$, given by (3.48) Multiply the first equation by $q^{-n\beta} = q^{-2}$, the second by $q^{-nx-1} = q^{-n\beta-k} = q^{-4}$, and let $\bar{X}(q^{-1}) = q^{-3}X_*(q)$. We then obtain equations in powers of q^{-1} only.

$$r(\beta_2+\beta_1 q^{-1}+q^{-2})R(q^{-1}) - (0.1+0.08q^{-1})\bar{X}(q^{-1}) = 0.1(1-q^{-1})(q^{-1}-1) . (q^{-1}-0.9)$$

$$r(\beta_2+\beta_1 q^{-1}+q^{-2})q^{-2}S(q^{-1}) + (1-0.9q^{-1})\bar{X}(q^{-1}) = q^{-1}(0.1q^{-1}+0.08) \ .$$

By considering terms with equal power of q^{-1}, a system of simultaneous equations, with block–Toepliz structure and with 10 equations and 8 unknowns, is obtained.

$$\left(\begin{array}{ccc|c|cccc} r\beta_2 & 0 & 0 & & -0.1 & 0 & 0 & 0 \\ r\beta_1 & r\beta_2 & 0 & 0 & -0.08 & -0.1 & 0 & 0 \\ r & r\beta_1 & r\beta_2 & & 0 & -0.08 & -0.1 & 0 \\ 0 & r & r\beta_1 & & 0 & 0 & -0.08 & -0.1 \\ 0 & 0 & r & & 0 & 0 & 0 & -0.08 \\ \hline & & & 0 & 1 & 0 & 0 & 0 \\ & & & 0 & -0.9 & 1 & 0 & 0 \\ & 0 & & r\beta_2 & 0 & -0.9 & 1 & 0 \\ & & & r\beta_1 & 0 & 0 & -0.9 & 1 \\ & & & r & 0 & 0 & 0 & -0.9 \end{array}\right) \begin{pmatrix} r_o \\ r_1 \\ r_2 \\ s_o \\ x_3 \\ x_2 \\ x_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} 0.09 \\ -0.28 \\ 0.29 \\ -0.10 \\ 0 \\ \hline 0 \\ 0.08 \\ 0.1 \\ 0 \\ 0 \end{pmatrix}$$

The (exact) solution, computed by least squares, is

$$\begin{aligned} R(q^{-1}) &= 1-0.08871q^{-1}+0.1210q^{-2} \\ S(q^{-1}) &= s_0 = 1.3617 \\ X_*(q) &= 0.4040+0.04945q+0.08q^2+0q^3 \ . \end{aligned}$$

In specific examples, the polynomial degrees may be lower than the values indicated by (3.48) and (3.50). This is evident here, where X_* actually only has degree 2. Note that, in general, the obtained $R(q^{-1})$ will be nonmonic, i.e. $r_0 \neq 1$.

The feedforward polynomial $Q(q^{-1})$ is obtained from the equation (3.34)

$$q^{-1}(0.2+0.4q^{-1})X_*(q) = r(1+\beta_1 q+\beta_2 q^2)Q(q^{-1}) + q(1-q^{-1})L_*(q)$$

with $X_*(q)$ from above, and with degrees $nQ = 2$, $nL = 1$ from (3.50). The solution is

$$Q(q^{-1}) = 1.8609+0.9750q^{-1}+0.6052q^{-2} \ ; \ L_*(q) = 0.2521-0.1675q \ .$$

Thus, the optimal regulator (3.30) is

$$\begin{aligned} u(t) &= 0.08871u(t-1) - 0.1210u(t-2) - 1.3617y(t) \\ &\quad -1.8609w(t) - 0.9750w(t-1) - 0.6052w(t-2) \end{aligned}$$

This regulator eliminates a unit step disturbances $w(t)$, after a small initial transient with peak value 0.16, without excessive input variations □

3.5 An LQG Self–tuner

The polynomial approach to adaptive LQG feedback control was first investigated a decade ago by Åström and Zhao–Ying and by Grimble [3] [17], for feedback only. LQG self–tuners based on state–space methods, using an iterative solution of the Riccati equation, have been proposed by different investigators. See e.g. [5], [8], [14] and [46]. There now exist commercially available adaptive LQG controllers, from the company First Control in Västerås, Sweden [6].

Compared to a pole placement algorithm, the "tuning knobs" of an LQG scheme are fewer and more closely related to design objectives. A frequency-weighted input penalty is more easy work with than a number of pole locations. Furthermore, stable common factors may occur in identified models. They cause no problems if the coupled equations (3.32) and (3.33) are used.

3.5.1 The adaptive algorithm

We will now present an adaptive LQG feedback–feedforward regulator in polynomial form [13]. It is based on results in the previous section. An explicit algorithm, based on the certainty equivalence principle, is considered. Models of $y(t)$ and $w(t)$ are updated recursively, using the Recursive Prediction Error Method (RPEM) [37]. The

[13]It has been developed and studied in [51]. A similar algorithm has been suggested in [23].

controller is redesigned periodically, assuming these models to be correct. Regulator polynomials are computed from Theorem 3.1. The regulator may be re–designed at each sample, or with larger intervals.

The regulator is designed to optimise the performance for zero set point, according the criterion (3.29). In the model structure (3.27)–(3.28), upper bounds on all polynomial degrees are assumed known, together with the unstable disturbance model factor $F(q^{-1})$. The input penalty ρ and the polynomials $V(q^{-1})$ and $W(q^{-1})$ in the criterion are user choices, given by the desiger.

The regulator, complemented with a servo filter, is summarised below.

1. Read new samples of $y(t), w(t)$ and a set-point $r(t)$.
2. Update models of $y(t)$ and $w(t)$ with the structure

   $$\hat{A}y(t) = \hat{B}u(t) + \hat{D}w(t) + \hat{C}\varepsilon_y(t) \quad ; \quad \hat{H}w(t) = \hat{G}\varepsilon_w(t)$$

   using two RPEM routines for single output systems.
3. Compute r and $\beta(q^{-1})$ from the spectral factorisation (3.31).
4. Solve for $R(q^{-1}), S(q^{-1})$ and $X_*(q)$ in (3.32),(3.33).
5. Calculate $Q(q^{-1})$ (and $L_*(q)$) from (3.34).
6. If needed, design a servo filter $T(q^{-1})/E(q^{-1})$.
7. Compute the control action:
   $RFu(t) = -(Q/\hat{G})w(t) - Sy(t) + (T/E)r(t)$
8. Shift all data vectors, and go to step 1.

3.5.2 Remarks on the algorithm

Step 2: Identification. The main model and the disturbance model are estimated by two separate recursive prediction error algorithms for single output systems. They constitute two Gauss–Newton algorithms, with the following structure

$$K(t) = P(t)\psi(t) = \frac{P(t-1)\psi(t)}{\lambda + \psi'(t)P(t-1)\psi(t)}$$

$$[\hat{\theta}(t) = \hat{\theta}(t-1) + K(t)[\varepsilon(t)]_{\mathrm{sat}}]_{\mathrm{proj}} \qquad (3.54)$$

$$P(t) = \frac{1}{\lambda}(P(t-1) - K(t)\psi'(t)P(t-1)) \ .$$

Here, $\varepsilon(t)$ is a scalar prediction error, $\psi(t)$ is the sensitivity vector $-\partial\varepsilon(t)/\partial\hat{\theta}$, and the matrix $P(t)$ is an approximation of the inverse Hessian. The updating of $P(t)$ should be implemented in factorised form, for numerical reasons. (The code in [37], which is based on UD-factorisation, has been used in the simulation study.) Above, λ is a forgetting factor $0 < \lambda \leq 1$. Time–varying systems and disturbance dynamics are tracked using $\lambda < 1$. The parameter vectors in the two algorithms are

$$\hat{\theta}_y = (\hat{a}_1 \dots \hat{a}_{na}\ \hat{b}_1 \dots \hat{b}_{nb}\ \hat{d}_0 \dots \hat{d}_{nd}\ \hat{c}_1 \dots \hat{c}_{nc})' \quad ; \quad \hat{\theta}_w = (\hat{h}_1 \dots \hat{h}_{nh}\ \hat{g}_1 \dots \hat{g}_{ng})'$$

respectively. The model polynomials $\hat{A}, \hat{H}, \hat{C}$ and $\hat{G}$ have fixed leading coefficients 1. Unknown and time–varying delays k and d are handeled by selecting degrees of $\hat{B}$ and $\hat{D}$ which cover the maximal expected delay. The terms $\hat{b}_1 \dots \hat{b}_{k-1}$ and $\hat{d}_0 \dots \hat{d}_{d-1}$ will then converge to zero[14].

As one–step prediction errors

$$\varepsilon_y(t) = y(t) - \varphi_{fy}(t)'\hat{\theta}_y(t-1) \quad ; \quad \varepsilon_w(t) = w(t) - \varphi_{fw}(t)'\hat{\theta}_w(t-1) \qquad (3.55)$$

are used, based on filtered regressor vectors

$$\varphi_{fy}(t) = \frac{F(q^{-1})}{N_1(q^{-1})}\varphi_y(t) \quad ; \quad \varphi_{fw}(t) = \frac{1}{N_2(q^{-1})}\varphi_w(t) \qquad (3.56)$$

where

$$\begin{aligned} \varphi_y(t) &= (-y(t-1) \dots - y(t-na)\ u(t-1) \dots u(t-nb) \\ &\quad w(t) \dots w(t-nd)\ \bar{\varepsilon}_y(t-1) \dots \bar{\varepsilon}_y(t-nc))' \end{aligned} \qquad (3.57)$$

$$\varphi_w(t) = (-w(t-1) \dots - w(t-nh)\ \bar{\varepsilon}_w(t-1) \dots \bar{\varepsilon}_w(t-ng))'$$

$$\bar{\varepsilon}_y(t) = y(t) - \varphi_{fy}(t)'\hat{\theta}_y(t) \quad ; \quad \bar{\varepsilon}_w(t) = w(t) - \varphi_{fw}(t)'\hat{\theta}_w(t) \ .$$

The use of prediction errors (3.55) instead of residuals $\bar{\varepsilon}_y$,$\bar{\varepsilon}_w$ as regressors would slow the convergence. Regressor filtering by (3.56), where $N_1(q^{-1})$ and $N_2(q^{-1})$ are stable polynomials, is utilised. The filtering of $\varphi_y(t)$ by $F(q^{-1})$ avoids biased estimates in the case of non–stationary or non–zero mean disturbances. With $N_1(q^{-1})$, the filter can be modified to improve the estimation accuracy in important freqency regions. In the simulations below, however, we use $N_1(q^{-1}) = N_2(q^{-1}) = 1$.

The sensitivity functions are obtained by filtering the regressor vectors (3.56):

$$-\partial\varepsilon_y(t)/\partial\hat{\theta}_y = \psi_y(t) = \frac{1}{\hat{C}(q^{-1})}\varphi_{fy}(t) \quad ; \quad -\partial\varepsilon_w(t)/\partial\hat{\theta}_w = \psi_w(t) = \frac{1}{\hat{G}(q^{-1})}\varphi_{fw}(t) \ . \qquad (3.58)$$

In (3.54), $[\,\cdot\,]_{\mathrm{proj}}$ represents stability monitoring and projection into stable regions of the polynomials $\hat{G}$ and $\hat{C}$. This is vital, not only because of the role of these polynomials in the control law, but also because they are used for generating the

[14]The algorithm would, however, have to be modified to cope with time delays above 10–15 sampling periods.

sensitivity vectors ψ_y, ψ_w above.

Furthermore, the function $[\varepsilon(t)]_{\text{sat}}$ in (3.54) represents a linear function, except for a dead–zone for small prediction errors, and a saturation for large $|\varepsilon(t)|$. While the former guards against estimator wind–up, the latter modification makes the adaptation more robust against rare but very large noise samples ("outliers"). Monitoring of the regressor energy has also been implemented as a precaution against identification based on insufficient data.

Step 6: Servo design. Since it is problematical to obtain good stochastic models of e.g. manually generated setpoints, we have *not* included servo desing in the optimization. Instead, we design a servo filter T/E by cancelling poles and stable zeros, so that the controlled system approximates a pre–specified response model $y_m(t) = (q^{-k}B_m/A_m)r(t)$. If the response model is not too extreme, this works well, but results in a rather high–order filter. Other approaches, such as including the servo design in the optimisation, are discussed in [51].

Step 7: Control computation. When appropriate, the regulator (3.52) or (3.53), based on equation (3.51), should be used.

Convergence. Global convergence of explicit LQG self–tuners can be demonstrated, under idealised conditions. See, for example, [12], [16] or [19]. In general, a linear model structure cannot be expected to describe the true system exactly. Mismodelling is inevitable, to some extent. A good estimate of $q^{-k}B/A$, in the frequency ranges where the input has significant energy, is needed to assure stability. See Example 3.4. Errors in the estimates of the transfer functions $q^{-d}D/A, C/A$ or G/H will affect the control performance, but they cannot cause instability. (Since the stability of $\hat{C}$ is monitored, pole placement in $\beta\hat{C}$ results in a stable system, if A and B are estimated correctly.)

Complexity. The computational burden of this algorithm is significantly higher than for simple direct self–tuners, such as generalised minimum variance control [13]. See Table 1. With the increasing speed of computers, this should be no significant restriction in most control applications. There is no need to recalculate the regulator at each sample. Steps 3-6 can be placed in a background process, which provides a new regulator every m'th sample. For $\lambda > 0.95$, the use of $m = 5 - 10$ results in only a small degradation of the adaptation transient when the system dynamics changes. (It has been shown by Shimkin and Feuer [48] that it may be important to update the regulator infrequently, to assure convergence.)

2:	Identification		$37n^2$	$+36n$
3:	Spectral factorisation (per iteration)		$3n^2$	$+3n$
4:	Feedback optimisation	$36n^3$	$+87n^2$	$+135n$
5:	Feedforward optimisation	$9n^3$	$+14n^2$	
7:	Control			8n

Table 3.1. The approximate number of mult–add operations required per sample, assuming all model polynomials to have equal degree n. A least squares solution is computed in Step 4.

3.5.3 Simulation examples

Example 3.2

Let $[1/(1-q^{-1})]v(t)$ be a square wave disturbance, with unit amplitude and period 60. It disturbs the system

$$(1-0.5q^{-1})y(t) = (b_2 + b_3 q^{-1})u(t-2) + (1+2q^{-1})w(t-1)$$

$$w(t) = \frac{1-0.3q^{-1}}{1-0.9q^{-1}}\left(\frac{1}{1-q^{-1}}v(t)\right) .$$

The polynomial $b_2 + b_3 q^{-1}$ changes from $1+0.1q^{-1}$ to $0.5+0.05q^{-1}$ at time 300. The LQG self-tuner, with correctly parametrized models, is applied. An input penalty $\rho = 0.5$ and $V = 1, W = 1-q^{-1}, F = 1$ is used. Thus, we use no integration. The forgetting factor λ is 0.98 in both RPEM algorithms. After an initial open loop identification period of 20 samples, the regulator quickly converges □

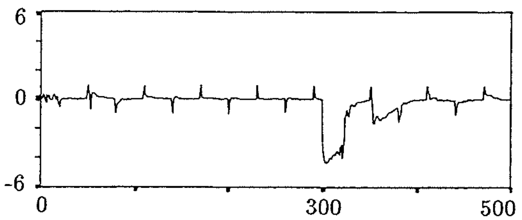

Figure 3.2: The controlled output $y(t)$ in Example 3.2. The disturbance $w(t)$ is cancelled almost completely, although the delay difference $k-d=1$ prevents perfect cancellation. At $t = 300$, the system gain is halved. At $t = 400$, the control performance has recovered to the off-line optimal one.

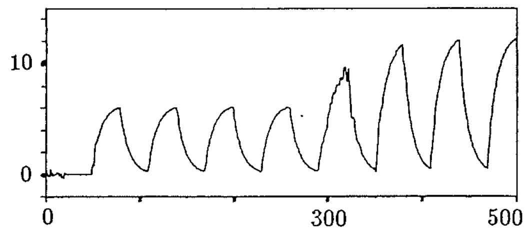

Figure 3.3: The input $u(t)$ in Example 3.2. When the system gain is halved at time 300, the regulator modifies itself, so that its gain is doubled.

Example 3.3

Consider the unstable and non-minimum phase system

$$(1-2q^{-1}+1.5q^{-2})y(t) = (1+2q^{-1}+2q^{-2})u(t-1) + (1+0.5q^{-1})w(t-2)$$

where $w(t)$ is white noise with standard deviation 0.1. As reference for the controlled output,

$$y_m(t) = \frac{0.7}{1-0.3q^{-1}}r(t)$$

was used, with $r(t)$ being a square wave. Adaptation, with a correctly parametrized model, and with $V = 1, W = 1, \rho = 0$, started at $t = 1$. The regulator had essentially converged to off-line optimal control after 30 samples. [15]. See Figure 3.4. Because the system is non-minimum phase, and the unstable part of B is not a factor of D, complete cancellation of the disturbance cannot be achieved □

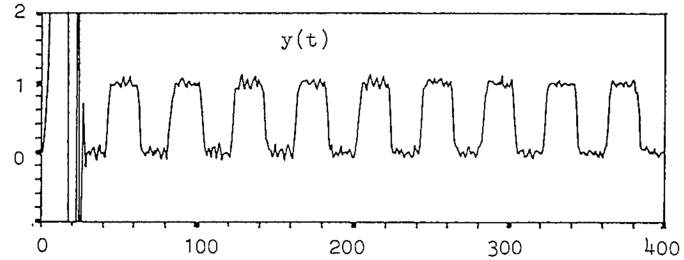

Figure 3.4: Control of the unstable non-minimum phase system in Example 3.3.

In further simulation studies, the adaptive algorithm has been found to behave very well in general. It has been compared to the Explicit criterion minimisation approach, cf. [55], which minimises the same criterion. Compared to that algorithm, the convergence rate of LQG self-tuners is much faster. See [51].

One (seldomly occuring) remaining problem is that over-parametrised models may contain *unstable* common factors of $\hat{A}$ and $\hat{B}$. The test of schemes, such as that of [34], to avoid this, is a problem for further research. A simple alternative can be used for the large class of systems which are open-loop stable or marginally stable. By using stability monitoring and projection on $\hat{A}$, in a similar way as is already done for $\hat{C}$ and $\hat{G}$, the risk of unstable common factors is eliminated.

3.6 User Choices Affecting the Robustness

The robustness against unmodelled dynamics of an LQG self-tuner is affected by properties of both the estimator and the control law. Simple considerations regarding the LQG control strategy, which in general improve the robustness of both off-line and self-tuning designs, are illustrated by the following example.

[15] Of course, it would not be advisable to use an adaptive regulator, without a prior model, during the start-up of such a system in practice; there is no guarantee that the signals behave acceptably in the transient phase. The instability at the beginning might cause too large input amplitudes, input saturation and thus divergence.

Example 3.4.

The system

$$(1 - 1.2q^{-1} + 0.52q^{-2})y(t) = q^{-2}(1 + 0.8q^{-1})u(t)$$
$$+ q^{-2}(1 - 0.2q^{-1})w(t) + (1 - 0.2q^{-1})n(t)$$

is affected by measurable and unmeasurable drifting stochastic disturbances

$$w(t) = w(t-1) + v(t) \;\; ; \;\; n(t) = n(t-1) + e(t) \; .$$

The white noises $v(t)$ and $e(t)$ have standard deviations 0.3 and 0.1, respectively. Thus, the largest disturbance is measurable, and $H_U = F = \Delta$.

The control error standard deviation was measured (after convergence) in simulation runs with four different self-tuners. Integrating regulators with the structure (3.52), with $r(t) = 0$ and $V = W = 1$, were used. The results are shown in Figure 3.5, as functions of the input penalty ρ. Curve (1) represents the performance of LQG feedback and feedforward. When $\rho \to 0$, the disturbance $w(t)$ is cancelled completely by the feedforward control action. When only feedback is used, curve (2) is obtained. The disturbances $w(t)$ and $n(t)$ are then treated as one unmeasurable noise. The performance is obviously degraded without disturbance measurement. Correctly parametrised models were used in these experiments. The performance in each case was indistinguishable from the off-line optimum.

Curve (3) and (4) result when an *underparametrised* $\hat{B}$ is used. (Degree 2 instead of 3, including the delay.) For input penalties $\rho \leq 1$, the closed loop is unstable.

The reason for this behaviour is explained by Figures 3.6 and 3.7. Figure 3.6 shows Bode magnitude plots of some under-parametrised models, obtained at the end of the simulation runs. Compare them with the true system. The high-frequency properties of the system are badly estimated. For low ρ, the regulators have large feedback gains at high frequencies, cf. Figure 3.7. (This is often the case for minimum variance regulators.) The combination of large feedback gain and an incorrect model at high frequencies leads to instability.

One way of reducing the high-frequency feedback gain is to modify the polynomial $C(q^{-1})$, used in (3.32)–(3.33). Instead of the estimate $\hat{C}$, a fixed polynomial $C_O = (1 - 0.5q^{-1})^2$ was used. This decreased the feedback high-frequency gain (cf. (4) in Figure 3.7) □

Based on the experience from this and other experiments, let us summarize some robustness-enhancing user-choices:

- By *increasing the input penalty* ρ *from zero*, the control signal variations, and the high-frequency gains of both feedback and feedforward filters, are reduced. Large

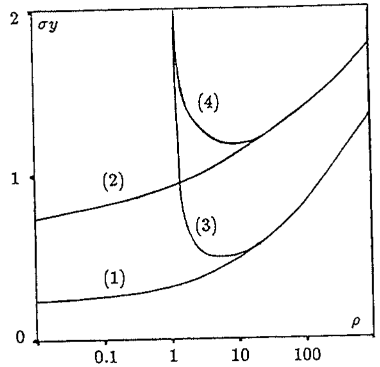

Figure 3.5: The output standard deviation σy in Example 3.4, as a function of the input penalty ρ.
(1): Feedback and feedforward. $\hat{B}$ of correct order 3.
(2): Feedback only. $\hat{B}$ of correct order 3.
(3): Feedback and feedforward. $\hat{B}$ of order 2.
(4): Feedback only. $\hat{B}$ of order 2.

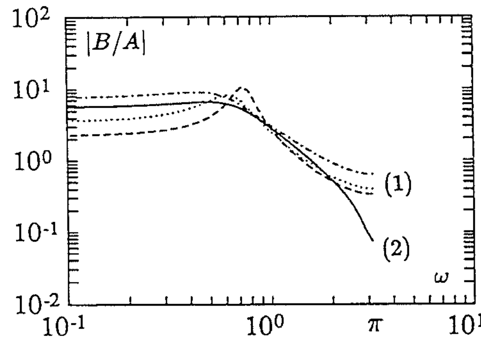

Figure 3.6: Bode magnitude plots of system and adapted models in Example 3.4.
(1): Transfer function magnitudes for some under-parametrized models.
(2): The true system.

reductions can often be achieved, with only a minor deterioration of the disturbance rejection. This increases the robustness against unmodelled high-frequency dynamics for open-loop stable systems. Problems with hidden inter-sample output oscillations are also avoided[16].

[16] "Hidden oscillations" are caused by pole placement on the negative real axis. The corresponding oscillative modes are unobservable in discrete time, at the sampling instants, but are evident in continuous time. See e.g. [4], Section 5.4. This often occurs in minimum variance control, with pole placement in BC, since B-polynomials of sampled data systems often have zeros on the negative real axis. See e.g. [4], Section 3.6.

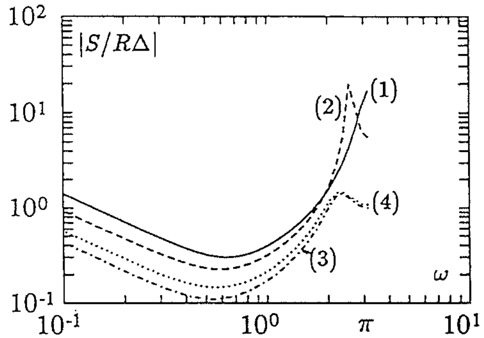

Figure 3.7: Transfer function magnitudes of feedback filters in Example 3.4.
(1): $\rho = 0$. (2): $\rho = 0.5$. (3): $\rho = 10$.
(4): $\rho = 0.5$, with pole placement in C_O.

- *Use of feedforward* can increase the stability robustness. This is possible when high disturbance rejection is required, and the main system disturbance is measurable. For example, consider Figure 3.5: if $\sigma y < 1$ is required, this could be attained, in the ideal case (2), by a high gain (low ρ) feedback. Instability would, however, result in the underparametrised case (4). With both feedback and feedforward, a low gain regulator ($\rho = 3 - 200$) can be used. It easily attains the required performance, also in the underparametrised case. Thus, when most of a disturbance can be eliminated by feedforward, the high-frequency gain of the feedback can be reduced. The feedback can more easily be designed to achieve robust stability. Also, more of the feedback control action can be allocated to robust performance issues, rather than to high (nominal) disturbance rejection.

- With LQG control, poles are placed at the zero locations of βC, cf. (3.49). The polynomial C corresponds to the observer dynamics in a state space formulation. Use of *a fixed prespecified observer polynomial* C_O, with $1/C_O$ being low-pass, has several advantages. First, uncertain estimates of C are avoided in the pole placement. The C–polynomials are *often hard to estimate* accurately, in particular if they have zeros close to the unit circle. Improved control performance may actually be achieved by not using them in the control law. Secondly, the estimated $\hat{C}$ may contain a factor close to $1-q^{-1}$. (That happens when regressors are differentiated in (3.56), but the disturbance $n(t)$ is not generated by a system with $F = 1-q^{-1}$.) It is obviously not desirable to force such a slow mode into the closed loop pole placement. Thirdly, unlike spectral factors β, we have no control over the zeros of C. A fixed C_O-polynomial may be used by the designer to obtain a safe and robustifying pole placement.

Appendix A. Conditions for, and verification of, finite cost in Section 3.3.

The conditions A3 and A4 are best understood if we consider drifting stochastic disturbances, generated by systems with poles at $z_j = 1$. (In other words, $\omega_j = 0$). To cancel these disturbances, we must have at least as many inputs as outputs. (If two nonstationary disturbances $w(t) = (w_1(t)\ w_2(t))'$ cause two outputs $y(t) = (y_1(t)\ y_2(t))'$ to drift in different directions, we could never hope to compensate for both of these drifts with just a scalar input $u(t)$.) Furthermore, the control action at the critical frequencies $\{\omega_j\}$ must not be blocked by zeros of the system BA^{-1}. These requirements are summarized by the condition A4.

To cancel a drifting disturbance, at least some components of the input vector $u(t)$ must also be drifting. The requirement A3 prevents such control modes, with infinite power, from appearing in the criterion. (Conditions analogous to A3 and A4 appear also in the scalar example of Section 3.4.)

Let us now verify that the optimal feedforward control law results in a finite criterion value. Using (3.17) and (3.21), the optimal criterion value is

$$\mathcal{E}(\mathrm{tr}(y_f y_f') + \mathrm{tr}(u_f u_f'))$$

where

$$y_f(t) = V(DE^{-1} - BA^{-1}\mathcal{R})GH^{-1}v(t) = V(DE^{-1}G - B\beta^{-1}QE_2^{-1})H^{-1}v(t)$$

$$u_f(t) = W\mathcal{R}GH^{-1}v(t) \ .$$

The transfer functions above have finite magnitude, except possibly at the frequencies $\{z_j\}$ which correspond to poles on the unit circle of GH^{-1}. They must be shown to be finite there also. For the input term $u_f(t)$, this is immediate from condition A3. That condition furthermore implies that

$$\beta_*\beta|_{z=z_j} = B_*V_*VB|_{z=z_j} \ .$$

The use of (3.23) gives

$$\beta^{-1}QE_2^{-1} = \beta^{-1}\beta_*^{-1}(B_*V_*VDG_2E_2^{-1} - L_*zH) \ .$$

Let us evaluate the transfer function from $v(t)$ to $y_f(t)$ at $z = \{z_j\}$. Using the two above relations and (3.20), it can be expressed as

$$V(DE^{-1}G - B\beta^{-1}\beta_*^{-1}(B_*V_*VDG_2E_2^{-1} - L_*zH))H^{-1}\Big|_{z=z_j} =$$

$$V(DE^{-1}G - B(B_*V_*VB)^{-1}B_*V_*VBB^{\dagger}DE^{-1}G)H^{-1} + VB\beta^{-1}\beta_*^{-1}L_*z\Big|_{z=z_j} =$$

$$V(DE^{-1}G - BB^{\dagger}DE^{-1}G)H^{-1} + VB\beta^{-1}\beta_*^{-1}L_*z\Big|_{z=z_j}$$

Since $BB^{\dagger} = I_p$, the first parenthesis is zero at $z = z_j$, so the first term cancels out. The second term is finite, since neither β^{-1} nor β_*^{-1} has poles on the unit circle. Thus, they can have no poles at z_j. Consequently, the non-stationary modes cancel at the output. This is, of course, a non-robust property, in that it requires perfect model knowledge at the critical frequencies $\{\omega_j\}$.

Appendix B. Verification of finite cost in Section 3.4

Let J_0 denote the cost (3.29) when the optimal regulator is applied. Using $R_1 = RF, S_1 = S, Q_1 = \beta Q$ and $P_1 = G\beta RF$ in (3.37), it can be expressed as

$$J_0 = \mathcal{E}(y_v(t) + y_e(t))^2 + \rho\mathcal{E}(z_v(t) + z_e(t))^2$$

where $WFu(t) \triangleq z_v(t) + z_e(t)$, $Vy(t) \triangleq y_v(t) + y_e(t)$ and

$$y_v(t) = \frac{VM}{\alpha G}\frac{G}{H_S H_U}v(t) \quad ; \quad y_e(t) = \frac{VFRC}{\alpha}\frac{1}{F}e(t)$$

$$z_v(t) = -\frac{WF(q^{-d}DSG + AQ)}{\alpha G}\frac{G}{H_S H_U}v(t) \quad ; \quad z_e(t) = -\frac{WFSC}{\alpha}\frac{1}{F}e(t)$$

Above, $M \triangleq q^{-d}DRFG - q^{-k}BQ$. Both $y_e(t)$ and $z_e(t)$ are stationary and have finite variance, since the cancellation of the unstable factor F is assumed to be exact.

The signal $y_v(t)$ has finite variance if (and only if) the unstable denominator factor H_U divides M. Consider the polynomial $r\beta_*M$. By using first (3.34) and then (3.32), it can be expressed as

$$\begin{aligned} r\beta_*M &= q^{-d}DFGr\beta_*R - q^{-k}Br\beta_*Q \\ &= q^{-d}DFGr\beta_*R - q^{-d}DFGq^{-k+1}BX_* + q^{-k+1}BCHL_* \\ &= q^{-d}DFG\rho WW_*F_*A_*C + q^{-k+1}BCH_S H_U L_* \end{aligned}$$

Since H_U is assumed to be a factor of DFW, it will be a factor of β_*M. But β_* has no zeros on the unit circle. So H_U, which has all its zeros on the unit circle, must be a factor of M. Consequently, $y_v(t)$ has finite variance.

Factors of H_U which are factors of WF are cancelled in the expression for $z_v(t)$ above. Common factors of H_U and D must, according to (3.34), also be factors of Q. (They cannot be factors of β_*, since β_* is assumed to have no zeros on the unit circle.) Thus, such factors are factors of $q^{-d}DSG + AQ$. Consequently, $z_v(t)$ is stationary, with finite variance, if H_U is a factor of WFD.

Bibliography

[1] A. Ahlén and M. Sternad, "Wiener filter design using polynomial equations", *IEEE Transactions on Signal Processing,* vol. 39, pp 2387–2399, 1991.

[2] A. Ahlén and M. Sternad, "Optimal filtering problems", in K. Hunt. ed. *Polynomial methods in Optimal Contol and Filtering.* Control Engineering Series, Peter Peregrinus, London, 1992.

[3] K. J. Åström and Z. Zhao–Ying, "A linear quadratic gaussian self–tuner," *Richerche di Automatica,* vol 13, pp. 106–122, 1982.

[4] K. J. Åström and B. Wittenmark, *Computer–Controlled Systems.* Prentice–Hall, Englewood Cliffs, NJ, second ed. 1990.

[5] G. Bartolini et. al.,"The ICOF approach to infinite horizon LQG adaptive control," *Richerche di Automatica,* vol 13, pp. 123, 1982.

[6] G. Bengtsson, "New regulators for industrial use," *Swedish Control Conference "Reglermöte"* , Linköping, Sweden, Oct. 1990.

[7] B. Bernhardsson and M. Sternad, "Feedforward control is dual to deconvolution." To appear in the *International Journal of Control,* 1992.

[8] R. R. Bitmead, M. Gevers and V. Wertz, *Adaptive Optimal Control; the Thinking Man's GPC.* Prentice Hall International, London, 1990.

[9] H. Bourles and E. Irving, "LQG/LTR: a polynomial approach," *European Control Conference ECC,* Grenoble, France, 1991.

[10] S. P. Boyd and C. H. Barratt, *Linear Controller Design: Limits of Performance.* Prentice–Hall, Englewood Cliffs, NJ, 1991.

[11] A. Casavola and E. Mosca, "Polynomial LQG regulator design for general system configurations," Proc. *30th CDC,* Brighton, England, pp 2307–2312, 1991.

[12] H. F. Chen and L. Guo, "Optimal adaptive control with consistent parameter estimates for ARMAX models with quadratic cost," *SIAM Journal on Control and optimisation,* vol 25, pp. 845–867, 1987.

[13] D. W. Clarke and P. J. Gawthrop, "Self–tuning control," *IEE Proceedings, Pt D,* vol 126, pp. 633-640, 1979.

[14] D. W. Clarke, P. P. Kanjilal and C. Mohtadi, "A generalized LQG approach to self-tuning control," *International Journal of Control*, vol 41, pp. 1509–1543, 1985.

[15] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, vol 12, pp. 457-465, 1976.

[16] L. Guo, "Robust stochastic adaptive control for non-minimum phase systems," *IFAC workshop on Robust Adaptive Control*, Newcastle, Australia. Preprints, pp. 185–189, 1988.

[17] M. J. Grimble, "Implicit and explicit LQG self-tuning controllers," *Automatica*, vol 20, pp. 661–669, 1984.

[18] M. J. Grimble, "Multivariable controllers for LQG self-tuning application with coloured measurement noise and dynamic cost weighting," *International Journal of System Science*, vol. 17, no. 4, pp. 543–557, 1986.

[19] M. J. Grimble, "Convergence of explicit LQG self-tuning controllers," *IEE Proceedings, Pt D*, vol 135, no 4, 1988.

[20] M. J. Grimble, "Two-degrees of freedom feedback and feedforward optimal control of multivariable stochastic systems," *Automatica*, vol 24, pp. 809-817, 1988.

[21] M. J. Grimble, "LQG optimal controller design for uncertain systems," *Proc. IEE, part D*, vol 139, pp. 21–30, 1992.

[22] M. J. Grimble and M. A. Johnsson, *Optimal Control and Stochastic Estimation.* Wiley, Chichester, 1988.

[23] K. J. Hunt, *Stochastic Optimal Control Theory with Application to Self-tuning Control.* Springer–Verlag, Berlin, 1989.

[24] K. J. Hunt and M. Šebek, "Optimal multivariable regulation with disturbance measurement feedforward," *International Journal of Control*, vol. 49, pp. 373–378, 1989.

[25] K. J. Hunt, M. Sebek and V. Kučera, "Polynomial approach to $\mathcal{H}_2$–optimal control: the multivariable standard problem," Proc. 30th CDC, Brighton, England, pp 1261–1266, 1991.

[26] J. Ježek and V. Kučera, "Efficient algorithm for matrix spectral factorization," *Automatica* vol. 21, pp. 663–669, 1985.

[27] V. Kučera, *Discrete Linear Control: The Polynomial Equations Approach.* Wiley, Chichester, 1979.

[28] V. Kučera, "Stochastic multivariable control: a polynomial equations approach", *IEEE Transactions on Automatic Control,* vol. 25, pp. 913–919, 1980.

[29] V. Kučera, "New results in state estimation and regulation," *Automatica*, vol. 17, pp. 745–748, 1981.

[30] V. Kučera, "The LQG control problem: A study of common factors," *Problems in Control and Information Theory*, vol 13, pp. 239-251, 1984.

[31] V. Kučera, "Diophantine equations in control," *European Control Conference ECC*, Grenoble, France, 1991.

[32] V. Kučera, *Analysis and Design of Linear Control Systems.* Academia, Prague and Prentice Hall International, London, 1991.

[33] V. Kučera and M. Šebek, "A note on stationary LQG control," *IEEE Transactions on Automatic Control*, vol 30, pp. 1242–1245, 1985.

[34] P. De Laminat, "On the stabilizability condition in indirect adaptive control," *Automatica*, vol 20, pp. 793-795, 1984.

[35] B. K. Lee, B. S. Chen and Y. P. Lin, "Extensions of linear quadratic optimal control theory for mixed backgrounds," *International Journal of Control*, vol 54, pp. 943–972, 1991.

[36] L. Ljung, *System Identification – Theory for the User.* Prentice–Hall, Englewood Cliffs, NJ, 1987.

[37] L. Ljung and T. Söderström, *Theory and Practice of Recursive Identification.* MIT Press, Cambridge, MA, 1983.

[38] J. M. Maciejowski, *Multivariable Feedback Design.* Addison Wesley, Reading, Mass, 1989.

[39] J. M. Mendel, *Optimal Seismic Deconvolution.* Academic Press, New York, 1983.

[40] M. Morari and E. Zafiriou, *Robust Process Control.* Prentice–Hall, Englewood Cliffs, NJ, 1989.

[41] E. Mosca and G. Zappa, "Matrix fraction solution to the discrete–time LQ stochastic tracking and servo problems," *IEEE Transactions on Automatic Control*, vol. 34, pp. 240–242, 1989.

[42] V. Peterka, "On steady–state minimum variance control strategy," *Kybernetika*, vol 8, pp. 219–232, 1972.

[43] V. Peterka, "Predictor–based self–tuning control," *Automatica*, vol 20, pp. 39-50, 1984.

[44] A. P. Roberts and M. N. Newmann, "Polynomial approach to Wiener filtering," *International Journal of Control*, vol. 47, pp. 681–696, 1988.

[45] H. H. Rosenbrock, *State–Space and Multivariable Theory.* Thomas Nelson and Sons, London, 1970

[46] C. Samson, "An adaptive LQ controller for non–minimum phase systems," *International Journal of Control*, vol 35, pp 1–28, 1982.

[47] M. Šebek, K. J. Hunt and M. J. Grimble, "LQG regulation with disturbance measurement feedforward," *International Journal of Control*, vol 47, pp. 1497-1505, 1988.

[48] N. Shimkin and A. Feuer, "On the necessity of 'block–invariance' for the convergence of adaptive pole–placement algorithm with persistently exciting input," *IEEE Trans. on Automatic Control*, vol 33, pp. 775-780, 1988.

[49] T. Söderström and P. Stoica, *System Identification.* Prentice Hall International, London, 1989.

[50] M. Sternad, "Disturbance decoupling adaptive control," *IFAC ACASP-86, Lund, Sweden.* Preprints, pp. 399-404, 1986.

[51] M. Sternad, *Optimal and Adaptive Feedforward Regulators.* PhD Thesis, Department of Technology, Uppsala University, Sweden, 1987.

[52] M. Sternad and T. Söderström, "LQG–optimal feedforward regulators," *Automatica* vol. 24, pp. 557–561, 1988.

[53] M. Sternad, "The use of disturbance measurement feedforward in LQG self–tuners," *International Journal of Control*, vol 52, pp. 579–596, 1991.

[54] M. Sternad and A. Ahlén, "A novel derivation methodology for polynomial–LQ controller design," *IEEE Transactions on Automatic Control*, vol 37, October 1992. Report UPTEC 90058R, Dept of Technology, Uppsala University.

[55] E. Trulsson and L. Ljung, "Adaptive control based on explicit criterion minimization," *Automatica*, vol 21, pp. 385-399, 1985.

[56] J. E. Weston and J. A. Bongiorno, "Extension of analytical desing techniques to multivariable feedback control systems," *IEEE Transactions on Automatic Control*, vol 17, pp. 613–620, 1972.

[57] V. A. Wolovich, *Linear Multivariable Systems.* Springer–Verlag, New York, 1974.

Chapter 4

Mixed H_2/H_∞ Stochastic Tracking and Servo Problems

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4.1 Introduction

Multiple objective control problems, involving different types of norms on closed-loop transfer functions, are motivated by tradeoffs among competing objectives, e.g. stability robustness versus performance. Well established results exist for performance optimization for given plant and disturbance models in an H_2-norm setting. On the other hand, H_∞ control is finalized to robustify closed-loop stability in the face of plant unstructured uncertainties. When both performance and robust stability are considered, mixed H_2/H_∞ optimization may turn out to be more physically justifiable.

Though a general solution to mixed H_2/H_∞ control seems hard to be found, in special cases the solution can be obtained by solving independently H_2 and H_∞ problems. In this chapter we address one of this special cases, viz. the minimax LQ stochastic servo and tracking problem.

The treatment is for multivariable linear discrete-time plants and stochastic exogenous signals (reference and disturbance) of finite variance, with possible uncertainty on the stochastic properties of the disturbances.

In our formulation the reference to be tracked is assumed to be known up to the generic time $t+\tau$, τ being an arbitrary positive or negative integer and t the current time. Then, the servo problem ($\tau > 0$) and the tracking one ($\tau \leq 0$) are unified.

One finds that a two-degrees-of-freedom (2DOF) controller is the optimal structure for this problem and that a suitable separation property exists between the feedforward and the feedback actions, in that both of them can be designed independently one from the other. In particular, the feedforward loop is optimized with respect to an H_2 criterion costing the output tracking error, while the feedback part is determined by solving an H_∞ mixed-sensitivity pure regulation (1DOF) problem.

The above approach can be used to add a model following capability to the above optimal control law. In fact, since the Model Matching (MM) problem presumes the existence of a reference signal to be tracked, the problem can be approached as an optimal servo or tracking problem.