Chapter 1

Summary and Introduction

1.1 Introduction

The present thesis will focus on the *estimation of signals* and on *feedforward control.* The aim is to present a methodology for model-based design, based on uncertain, linear and time-invariant dynamic models. The resulting method will be based on a probabilistic description of the uncertainty in the assumed dynamics and on the minimization of averaged mean square error, or $\mathcal{H}_2$, criteria.

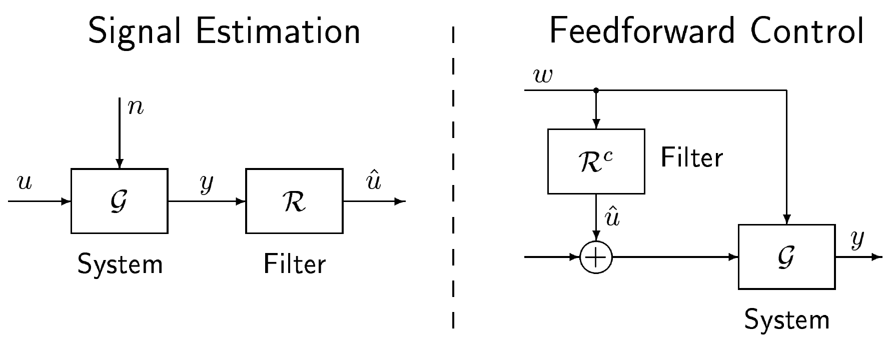

Figure 1.1: Two types of problem formulations considered in the present thesis: The estimation of discrete-time signals from noisy data (left) and feedforward control based on measurable discrete-time signals (right).

The two considered types of problems, illustrated in Figure 1.1, can be characterized as follows:

- **Signal Estimation.** Based on discrete-time measurements, $y(k)$, we seek an estimate, $\hat{u}(k)$, of a desired signal, $u(k)$. The desired signal passes through, and is affected by, a linear time-invariant dynamic system $\mathcal{G}$. Furthermore, disturbances, $n(k)$, are corrupting the measurements. To obtain the estimate, a linear discrete-time filter, $\mathcal{R}$, is to be designed.

The problem of signal estimation appears is numerous types of applications, of which the determination of temperatures, velocities and accelerations constitute some examples. Another type of application is mobile communications, where messages are to be retrieved from received signals.

- **Feedforward Control.** Measurable disturbances or known command signals, $w(k)$, do here affect a linear time-invariant dynamic system $\mathcal{G}$. Through the use of a linear discrete-time feedforward filter, $\mathcal{R}^c$, the effects of $w(k)$ on the output, $y(k)$, is to be controlled by an input signal $\hat{u}(k)$.

  Feedforward control includes disturbance measurement feedforward, where the effect of a measurable disturbance, $w(k)$, is to be reduced or even cancelled. The problem formulation also includes servo design problems in which the output from the system has to follow a command signal $w(k)$. In some applications, the filter $\mathcal{R}^c$ is simply a gain factor or constant matrix, while in other situations it is a complicated dynamic system.

The design of filters or regulators could be based directly on experiment data. Examples of such *direct data-based* methods are direct adaptive algorithms for filtering and control, as well as experiment-based iterative adjustment of filters with fixed structure. Alternatively, one can choose an *indirect model-based* methodology. Dynamic models describing the relations between signals are then obtained by first principles, through system identification, or by other means. The filter or controller is then computed based on these models. Model-based methods include many schemes for off-line optimization of filters, as well as for indirect adaptive filtering and control. The present thesis will focus on model-based methods, which utilize tools from statistical signal processing.

In the present thesis, discrete-time (stationary) stochastic processes are used to model signals. In such models, illustrated by Figure 1.2, the input signals $e_u(k)$, $e_n(k)$ and $e_w(k)$ are white random vectors, and $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{W}$ are linear discrete-time stable dynamic models selected to give the output signal vectors $u(k)$, $n(k)$ and $w(k)$ desired properties. These properties should be adjusted to describe the *a priori* knowledge about the system under consideration. The stochastic models are then used for the design. In, for example, a model-based design of a signal estimator, the resulting estimator

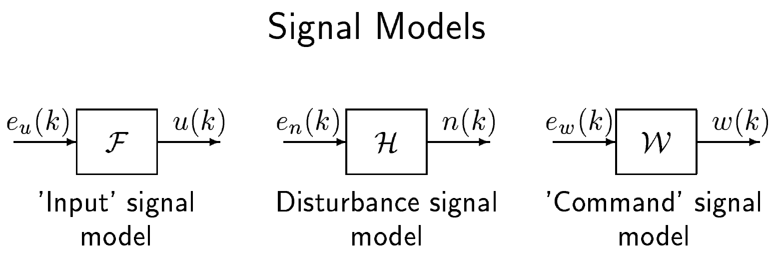

Figure 1.2: Discrete-time, linear, stochastic models for the signals $u(k)$, $v(k)$ and $w(k)$ introduced in the problem formulations described in Figure 1.1.

$\mathcal{R}_1$ will be a function of the *a priori* information and on the criterion J_1 to be minimized, i.e.

$$\mathcal{R}_1 = f\left(\mathcal{F}, \mathcal{G}, \mathcal{H}, J_1\right). \tag{1.1}$$

A problem with model-based design is that the inevitable modelling errors constitute a potential source of performance degradation; If the models are not accurate, then the performance of the filter, when applied to the actual system, will differ from what could be expected.

To investigate the effects of model imperfections, and to reduce these effects, models have to be complemented by quantitative information about *model uncertainty*. Graphically, this can be illustrated as shown in Figure 1.3.

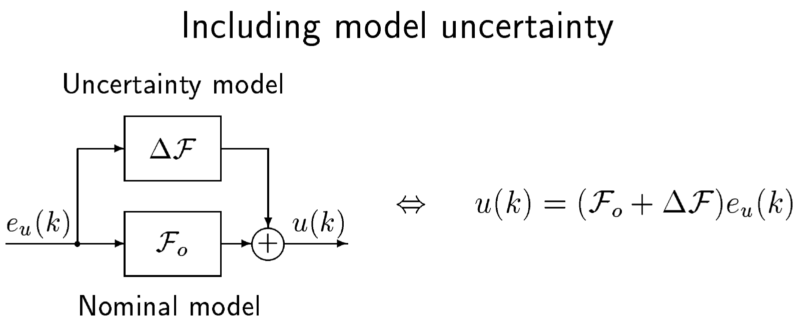

Figure 1.3: An example of a more complete signal modelling. A nominal model, $\mathcal{F}_o$, is here complemented by a set of dynamic systems, an uncertainty model, $\Delta\mathcal{F}$, which reflects the range of possible deviations from the nominal model.

If the relative differences between design models and actual systems are small, then the model errors can be neglected. On the other hand, if the differences are very large, then the accuracy of the models has to be improved through, for example, additional modelling, identification or adaptation. In between these two extremes, there exists a wide range of model error sets, for which the performance of a filter could be improved by taking knowledge about the size of the uncertainty into account in the design. Formally, we may then replace (1.1) by the computation of a modified filter, $\mathcal{R}_2$,

$$\mathcal{R}_2 = g\left(\mathcal{F}, \Delta\mathcal{F}, \mathcal{G}, \Delta\mathcal{G}, \mathcal{H}, \Delta\mathcal{H}, J_2\right) \ , \tag{1.2}$$

where J_2 is now a performance criterion which is to be minimized also with respect to the uncertainty models $\Delta\mathcal{F}$, $\Delta\mathcal{G}$ and $\Delta\mathcal{H}$. This is the aim of a method for *robust design.* To summarize;

The present thesis is devoted to the model-based design of linear discrete-time filters and feedforward regulators, in which uncertainty in dynamic design models is explicitly taken into account.

The design methodology will be based on a *probabilistic description of model errors* in linear dynamic models. Thus, in Figure 1.3, the model uncertainty represented by $\Delta\mathcal{F}$ will be considered as a stochastic entity.

The primary goal of the present chapter is to give an introduction to the proposed design philosophy and the obtained results. We begin with an example, which illustrates how the performance of a signal estimator may be degraded when the true system does not coincide with the nominal model used in the design.

Example 1.1 *An introductory example*

Consider the following linear and discrete-time stochastic system

$$y(k) = B(q^{-1}, \rho)\, u(k) + v(k) \ ,$$

where the scalar signal $y(k)$ is measurable and where the scalar signal $u(k)$ is an autoregressive stochastic process

$$u(k) = \frac{1}{D(q^{-1})}\, e(k) \ .$$

Here, $\{e(k)\}$ and $\{v(k)\}$ are assumed to be mutually uncorrelated, zero-mean, stationary, and white noise sequences with known variances $\sigma_e^2 = 1$ and $\sigma_v^2 = 0.01$, respectively. Using the backward shift operator, $q^{-1}y(k) = y(k-1)$, the transfer operators in the model above are specified by the following two polynomials

$$\begin{aligned} B(q^{-1}, \rho) &= 1 - (1.60 + \rho)\, q^{-1} + 0.65\, q^{-2} \\ D(q^{-1}) &= 1 - 0.50\, q^{-1} \ . \end{aligned}$$

Above, $B(q^{-1},\rho)$ represents an imperfectly known transducer, characterized by the unknown parameter ρ, whereas $1/D(q^{-1})$ is a known autoregressive model of the input signal $u(k)$. The objective is now to find the linear filtering estimate, $\hat{u}(k|k)$ (based on measurements up to time k), of the input signal, $u(k)$, which minimizes the variance of the estimation error

$$\varepsilon(k) = u(k) - \hat{u}(k|k) \ .$$

In other words, based on the nominal model, represented by $B(q^{-1},\rho)$ with $\rho = 0$, $D(q^{-1})$, $\sigma_e^2 = 1$ and $\sigma_v^2 = 0.01$ and based on measurements $y(s)$, $s \leq k$, a stable and linear estimator,

$$\hat{u}(k|k) = \mathcal{R}(q^{-1})\, y(k) \ ,$$

is sought. See Figure 1.4.

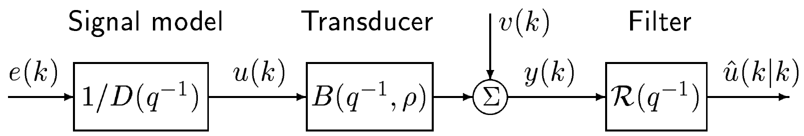

Figure 1.4: The problem setup of Example 1.1. The input $u(k)$ is to be estimated in spite of a parametric uncertainty in the second order FIR transducer model $B(q^{-1},\rho)$.

The transfer function $\mathcal{R}(q^{-1})$ is to be designed so that the mean square error criterion

$$J = E\left(u(k) - \hat{u}(k|k)\right)^2 \tag{1.3}$$

is minimized. Here, $E(\cdot)$ denotes expectation taken with respect to the stochastic noise sequences.

If the model corresponds to the true system, then the solution to this problem can be obtained by using a nominal Wiener or Kalman filter design. Straightforward use of design equations from [2] results in the following nominal Wiener filter

$$\mathcal{R}_n = \frac{0.8904}{1 - 1.5411\, q^{-1} + 0.6096\, q^{-2}} \ .$$

Figure 1.5 shows how the mean square estimation error (MSE) (1.3) is affected when $\mathcal{R}_n$ is applied to systems with different values of the uncertain parameter ρ (dashed curve). The figure also shows the variance of the input

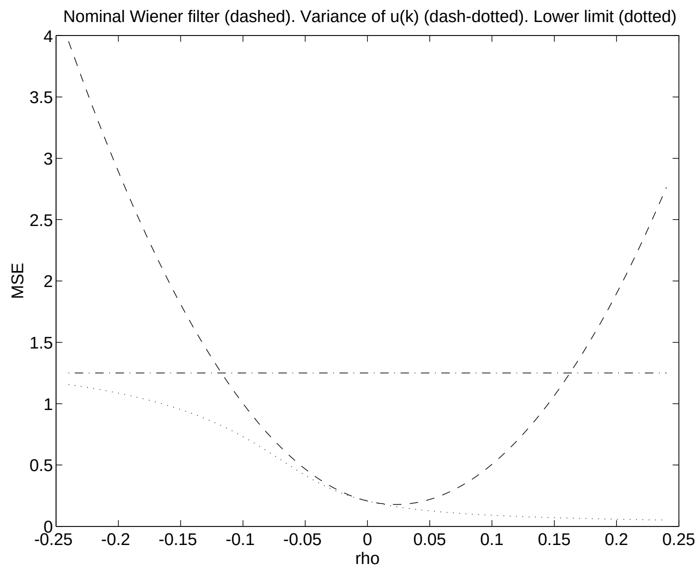

Figure 1.5: The estimation error variance (MSE) (1.3) as a function of the uncertain parameter ρ (= rho). The plot shows the performance obtained with the nominal Wiener filter (dashed), the lower limit, obtained if the correct model is used for every value of ρ (dotted), and the performance obtained when using the trivial "zero-estimate" $\hat{u}(k|k) \equiv 0$ (dash-dotted).

signal $u(k)$ (dash-dotted curve), and also the performance that could be obtained for the correct value of ρ is illustrated (dotted curve).

From Figure 1.5 it can be concluded that the performance of the nominal filter, $\mathcal{R}_n$, is rather sensitive to the actual value of ρ.[1] The performance when ρ is outside of the range $\rho \in (-0.12, 0.16)$ is clearly unacceptable in the sense that it is worse than the "trivial" estimate $\hat{u}(k|k) \equiv 0$ (dash-dotted), which gives $\varepsilon(k) = u(k)$. If parameter deviations of the order of magnitude displayed by Figure 1.5 are expected, then there would be a need either to estimate ρ or design a robust filter. A robust design would then take the model uncertainty, here the range of the variation of ρ, into account. □

[1] A high signal-to-noise ratio (SNR $= 10\log(Eu^2(k)/Ev^2(k)) \approx 21$dB) has deliberately been selected in the present example to emphasize the model mismatch.

1.2 Issues in robust filter design

Robustness is, in itself, an ill-defined concept. It is common to distinguish between *stability robustness* (maintaining stability in the presence of uncertainty or variations in system properties) and the more ambitious goal of *performance robustness*. The latter concept often means that a filter should be designed so that a pre-specified level of performance can be guaranteed for a specified type of system variation or model uncertainty. In other problem formulations, improving the performance robustness means that the average performance of the design shall be improved. It should be noted that there are always performance limitations in linear filtering, see, for example, the work [33] of Goodwin *et al.*

To obtain robust designs, we must therefore specify the type of robustness and the type of uncertainty, or system variations, for which robustness should be improved. There are numerous possible causes for the misbehaviour of a signal processing system. For a given problem, we cannot expect to guard against all different kinds of uncertainties with a single design tool. Below, four types of uncertainties are outlined, which may not all be amendable by using the same type of robust design.

1. **Parametric uncertainty.** Although the degree of a linear system is sometimes well known, some of the parameters are here assumed to be uncertain to a specified extent. This was the case in Example 1.1 above. Both the known dynamics and the uncertain parameters are sometimes allowed to be time-varying. Parametric uncertainty arises, for example, when a non-linear system is linearized around operating points or state trajectories, with differing linearized dynamics. Noise-induced uncertainty in identified models (represented by the parameter covariance matrix) also constitutes a parametric uncertainty. Timing jitter, i.e. small variations in the sampling rate, can be represented by a parametric time-varying uncertainty in a sampled-data model.

   Methods designed to attain performance robustness in the presence of parametric uncertainty aim at reducing the influence of, or even decouple, the uncertain parameters from the signals affecting the performance. In feedback systems, the use of a high gain feedback is perhaps the most well-known tool for reducing sensitivity with regard to parametric uncertainty.

2. **Non-parametric uncertainty.** A high order dynamic system of unknown structure and degree is often approximated by a low order linear model of a specified degree. This situation is denoted *un-*

dermodelling, and the differences between the system and the model represent a non-parametric, or *unstructured*, uncertainty. Identified models mostly suffer from both parametric and non-parametric uncertainty, see e.g. Ljung [72]. The question of how to make good use of any available knowledge about the unstructured model errors is, in general, difficult to answer. Hard or probabilistic bounds on the frequency responses are often used. Such concepts may, however, be of dubious value if the unmodelled dynamics is inherently nonlinear.

Robust performance of feedback control systems in the presence of both parametric and non-parametric uncertainty is an active research area with, as yet, no fully satisfactory solutions[2]. The problem has been studied very little with regard to open-loop designs.

3. **Uncertain statistics and outliers.** If a filter is designed to be optimal for Gaussian noise and/or signals, then the performance of the filter may deteriorate significantly in the presence of signals with non-Gaussian statistics. The effect of outliers (noise samples with large magnitudes) on estimates obtained by Kalman filters or by RLS/LMS adaptation laws[3] is another example.

   Filters and detectors can be robustified with regard to uncertain statistics by applying nonlinear transformations on measured signals or on innovations signals. Various systematic schemes exist for designing such nonlinearities, for given assumed classes of signal distributions. Huber [49] performed the pioneering work in this field. See also the survey paper [59] by Kassam and Poor, the paper [76] by Martin and Mintz or [100] by Stahel. The simplest type of nonlinearity, effective against noise outliers, is a limiter. It usually operates on the residual signals in Kalman filters, Wiener filters and adaptation laws.

4. **Abrupt structural or parametric changes.** Such changes may be due to the action of control logic, sudden changes in operating conditions or sensor failures, among other causes. Filtering performance can sometimes be rendered insensitive to these phenomena by designing a set of different filters, each attuned to a specific situation or failure mode. An estimator or change-detection device is required for

[2] One can, for example, note that a high gain feedback, designed to reduce the influence of a parametric uncertainty, may result in instability if unstructured model errors are present and not accounted for. The best approach in this context could be the μ-method, i.e. the use of the structured singular values, described by Doyle *et al.*, see [28], [29], [101], [95]. See also Maciejowski [73].

[3] RLS and LMS stand for Recursive Least Squares and Least Mean Squares, respectively.

selecting the output from one of the filters or from a linear combination of the filter outputs. See, for example, the work by Lainiotis [66]. Multiple redundant sensors, actuators and filters can also be used to increase the reliability of systems for signal processing and control.

The way in which the model uncertainty is described, i.e. the *error modelling*, is a central part of any robust design method. Error models are, by necessity, imprecise; the exact modelling of uncertain dynamics would be a contradiction in terms. A requirement for elaborate error models, and a large engineering effort in obtaining them, would furthermore reduce the attractiveness of a robust design methodology. On the other hand, crude error models, which include a range of possible dynamics much wider than the range actually encountered, could result in conservative, and possibly useless, designs.

It can be difficult to ascertain the *stability* of an interconnected system in cases where the dynamics in some subsystems are partly unknown. The study of robust stability is a central issue in the analysis of feedback control systems. Stability can also be problematic in open-loop estimation and feedforward control problems. If not all subsystems are guaranteed to be stable within the whole uncertainty set, the differences between signals and their estimates may grow without bounds. This is called divergence in the literature on Kalman filtering.

In the present thesis, we will use a probabilistic approach to error modelling. We will not aim for guaranteed performance, but will instead strive to minimize the average performance within the uncertainty set. This problem formulation not only seems to be a good candidate of providing simplicity and generality, but also helps to avoid conservative designs.

We will consider the design of discrete-time, linear, robust filters. The filters are designed to be applied on open-loop, multivariable[4] discrete-time problems such as signal estimation, deconvolution, equalization, state estimation and feedforward control. The performance criterion of interest is the average mean square estimation error, or the average $\mathcal{H}_2$-norm of transfer functions. (The criterion will be defined in the next section.) The robustness of interest concerns *parametric or non-parametric, time-invariant uncertainties in the dynamics of linear design models.*

A large class of practically important problems, without feedback connections and with all uncertain systems assumed to be stable, is considered.

[4]Multivariable, in this context, refers to systems with multiple inputs and multiple outputs, also called MIMO systems.

In such situations, the non-existence of divergent estimates can be assured in a straightforward way. An interesting topic could also be to use the resulting linear, robust filters in schemes for obtaining also robustness against noise outliers or in filter-banks. The design of such nonlinear algorithms is, however, beyond the scope of the present thesis.

1.3 Stochastic uncertainty modelling and averaged $\mathcal{H}_2$-design

The approach to robust design to be presented in this thesis will be based on the following assumption of model structure, and on the following selection of optimization criterion:

- A set of (true) dynamic systems is assumed to be well described by a set of discrete-time, linear, stable and time-invariant transfer function matrices
$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_o + \Delta\boldsymbol{\mathcal{F}} \; . \tag{1.4}$$
We shall call such a set of models an *extended design model.* Above, $\boldsymbol{\mathcal{F}}_o$ represents a stable *nominal model*, while an *error model* $\Delta\boldsymbol{\mathcal{F}}$ describes a set of stable, linear and time-invariant transfer functions, parameterized by stochastic variables. The stochastic variables enter linearly into $\Delta\boldsymbol{\mathcal{F}}$.

- A single, robust, linear, stable, discrete-time filter is to be designed for the whole class of possible systems, modelled by $\boldsymbol{\mathcal{F}}$. Robust performance is obtained by minimizing the *averaged* mean square estimation error criterion
$$J = \text{trace}\bar{E}\, E\left(\varepsilon(k)\, \varepsilon(k)^*\right) \; . \tag{1.5}$$
Here, $\varepsilon(k)$ is the estimation error vector, whereas $E(\cdot)$ and $\bar{E}(\cdot)$ denote expectation with respect to stochastic noise sequences and the stochastic variables parameterizing the error model $\Delta\boldsymbol{\mathcal{F}}$, respectively.

There are many conceivable alternatives to the criterion (1.5). The mean square error could be replaced by another function. One example is the consideration of the time-domain ℓ_1 criterion; see, for example, Dahleh and Diaz-Bobillo [23], Dahleh [22] or Mendlovitz [78]. Much recent interest has been focused on the use of an $\mathcal{H}_\infty$ criterion instead of the MSE, or $\mathcal{H}_2$ criterion, see e.g. de Souza *et al.* [27], Nagpal and Khargonekar [81], Grimble [37], Grimble and El Sayed [39], Shaked and de Souza [92], Shaked and Theodor [93], Mangoubi *et al.* [74] or Xie *et al.* [115]. Instead of considering the *average* performance, one could instead choose to minimize the *guaranteed* performance for any system within the uncertainty set, see Section 1.4.

There are several reasons to consider the averaged MSE, or averaged $\mathcal{H}_2$-norm criterion:

- As opposed to an $\mathcal{H}_\infty$ design, in which the filters are entirely determined based on "worst case" signal spectra, the criterion (1.5) takes not only the range of the uncertainties in signal models into account but their likelihood as well.

- We believe that designers are often more interested in the average performance than the guarantee that the performance is bounded by (possibly very conservative) limits, determined by situations that might hardly ever occur in reality.

- It is our opinion that an approach to robust performance design based on the extended design models (1.4) and the criterion (1.5) results in a simpler, and less computationally demanding, design compared with most alternatives known at present.

We selected a design criterion averaged with respect to model uncertainties; thus, there is a need to stochastically describe these uncertainties. The averaged mean square error has been used as criterion previously in the literature, in particular by Chung and Bélanger [21], Speyer and Gustafson [99] and Grimble [36]. These works were based on assumptions of small parametric uncertainties in continuous-time models and on series expansions of uncertain parameters. In the present thesis, we suggest the use of the criterion (1.5) together with a special description of the error model $\Delta\boldsymbol{\mathcal{F}}$ in (1.4). In our approach, the error model is assumed to be a product of two factors,

$$\Delta\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_1 \Delta\boldsymbol{F} \ ,$$

where the transfer function elements of $\boldsymbol{\mathcal{F}}_1$ are fixed while $\Delta\boldsymbol{F}$ is a polynomial matrix with stochastic variables as coefficients. Such models, which can be obtained in many ways, can describe non-parametric uncertainty and under-modelling as well as parametric uncertainty. The proposed error models also relate to the stochastic embedding concept presented by Goodwin and co-workers in [32], [31]. It should be noted that, in the design methods to be derived, no particular distribution for the extended design model is required. Only the second order moments will be utilized in the design.

Example 1.1 *continued. An extended design model for the transducer*

We now assume that the transducer $B(q^{-1}, \rho)$, see Figure 1.4, belongs to a *set* of transducers parameterized by ρ. A particular value of ρ represents a particular transducer within the set. We assume that the average value of ρ is equal to zero. An extended design model can now be formulated as

$$B_m(q^{-1}) = B_o(q^{-1}) + \Delta B(q^{-1})$$

where

$$\begin{aligned} B_o(q^{-1}) &= b_o - b_1\, q^{-1} + b_2\, q^{-2} \\ &= 1 - 1.60\, q^{-1} + 0.65\, q^{-2} \\ \Delta B(q^{-1}) &= \Delta b_1\, q^{-1} \ . \end{aligned}$$

Above, the nominal model $B_o(q^{-1})$ is intended to capture the average behaviour of the set of transducers, while $\Delta B(q^{-1})$ should reflect the variations within the set. In the error model, $\Delta B(q^{-1})$, Δb_1 is used in place of ρ to distinguish the model set from the system set. The stochastic coefficient Δb_1 has the same variance as that assumed for ρ. □

The optimization of linear, time-invariant filters to minimize the mean square error criterion is known as *Wiener filtering*, and originates in work by Wiener [112]. The book [62] by Kučera has inspired several research groups to develop a polynomial approach to Wiener filtering and control problems. In [2], Ahlén and Sternad presented a new technique for constructively deriving polynomial matrix design equations for Wiener filters. An advantage of using a polynomial matrix approach is that a direct inspection of the filter expression readily reveals important properties of the design. Furthermore, efficient numerical algorithms are available for solving the design equations; see, for example, Kučera [62] and also Appendices A and B of this thesis.

In the present thesis the ideas presented in [2] will be used to find a polynomial matrix solution to the *averaged* criterion (1.5). Extended design models (1.4) will then be represented by using matrix fraction descriptions. The design equations for obtaining the robust multivariable Wiener filter will be of the same type as the ones used for a nominal design: polynomial matrix coprime factorizations, a polynomial matrix Diophantine equation and a polynomial matrix spectral factorization. As compared to the nominal design, see, for example, [3], [4] or [39], the robust design requires only straightforward additional calculations to take the assumed uncertainty into account. As an illustration, Example 1.1 is complemented by a robust design.

Example 1.1 *continued. A robust filter design*

The aim is now to find a robust estimator for the input signal $u(k)$, see Figure 1.4. By using the polynomial approach of e.g. [1] or [2], the nominal design equations consist of the spectral factorization,

$$\begin{aligned}\beta_n(e^{-\jmath\omega})\beta_n(e^{\jmath\omega}) &= \sigma_e^2 B_o(e^{-\jmath\omega})B_o(e^{\jmath\omega}) + \sigma_v^2 D(e^{-\jmath\omega})D(e^{\jmath\omega}) \\ &= 0.65\, e^{-\jmath 2\omega} - 2.645\, e^{-\jmath\omega} + 3.995 - 2.645\, e^{\jmath\omega} + 0.65\, e^{\jmath 2\omega},\end{aligned}$$

and the Diophantine equation,

$$B_o(q) = Q_n(q^{-1})\beta_n(q) + qL_n(q)D(q^{-1}) \;, \tag{1.6}$$

which are to be solved with regard to the polynomials $\beta_n(e^{-\jmath\omega})$, $Q_n(q^{-1})$ and $L_n(q)$, respectively. Based on these equations, the nominal filter is obtained as

$$\mathcal{R}_n(q^{-1}) = \frac{Q_n(q^{-1})}{\beta_n(q^{-1})} = \frac{0.8904}{1 - 1.5411\, q^{-1} + 0.6096\, q^{-2}} \;.$$

The stable polynomial $\beta_n(e^{-\jmath\omega})$ represents the numerator of an innovations representation of the measurement $y(k)$, which has the spectral density

$$\Phi_y(e^{\jmath\omega}) = \frac{\beta_n(e^{-\jmath\omega})\beta_n(e^{\jmath\omega})}{D(e^{-\jmath\omega})D(e^{\jmath\omega})}\, \Phi_\epsilon(e^{\jmath\omega})$$

where the spectral density, $\Phi_\epsilon(e^{\jmath\omega})$, of the white innovations sequence $\epsilon(k)$ has been normalized to one.

Assume now that the coefficient modelled by b_1 can be expected to deviate by no more than 15%. For a robust design minimizing (1.5) we can then select to model the uncertain parameter ρ by using a stochastic variable Δb_1, being uniformly distributed within a 15%-interval around zero, i.e. $\Delta b_1 \in \mathcal{U}(-0.24, 0.24)$. The variance is thus assumed to be $\bar{E}(\Delta b_1^2) = 0.0192$. The minimization of the criterion (1.5) results in a modified, averaged spectral factorization equation, which provides the denominator polynomial of the robust filter. Note that since Δb_1 is assumed to be independent of the noise processes $v(k)$ and $e(k)$, the average, with respect to Δb_1, of the output spectral density is given by

$$\bar{E}\left(\Phi_y(e^{\jmath\omega})\right) = \frac{\beta_r(e^{-\jmath\omega})\beta_r(e^{\jmath\omega})}{D(e^{-\jmath\omega})D(e^{\jmath\omega})}$$

where

$$\begin{aligned}\beta_r(e^{-\jmath\omega})\beta_r(e^{\jmath\omega}) &= \sigma_e^2 B_o(e^{-\jmath\omega})B_o(e^{\jmath\omega}) + \sigma_e^2 \bar{E}\left(\Delta B(e^{-\jmath\omega})\Delta B(e^{\jmath\omega})\right) \\ &\quad + \sigma_v^2 D(e^{-\jmath\omega})D(e^{\jmath\omega}) \\ &= 0.65\,e^{-\jmath 2\omega} - 2.645\,e^{-\jmath\omega} + 4.0142 - 2.645\,e^{\jmath\omega} + 0.65\,e^{\jmath 2\omega} .\end{aligned} \tag{1.7}$$

The only change in the Diophantine equation (1.6), which will be evident from Chapter 4, is the replacement of β_n by β_r:

$$B_o(q) = Q_r(q^{-1})\beta_r(q) + qL_r(q)D(q^{-1}) \ .$$

The resulting robust Wiener filter is then given by

$$\mathcal{R}_r(q^{-1}) = \frac{Q_r(q^{-1})}{\beta_r(q^{-1})} = \frac{0.6408}{1 - 1.3702\,q^{-1} + 0.5076\,q^{-2}} \ .$$

Here, the only difference between the nominal and robust designs is the calculation of the averaged "double-sided" polynomial $\bar{E}\,(\Delta B(e^{-\jmath\omega})\Delta B(e^{\jmath\omega}))$, which appears in the averaged spectral factorization (1.7). In the present example, we readily obtain

$$\bar{E}\,(\Delta B(e^{-\jmath\omega})\Delta B(e^{\jmath\omega})) = \bar{E}(\Delta b_1^2) \ = \ 0.0192 \ .$$

In Chapter 2 it is described how the corresponding calculations can be conducted in a multivariable setting. The performance of the robust filter when applied to different possible systems is displayed in Figure 1.6. The significant performance difference between robust and nominal designs results from the, seemingly, small increase from 3.995 to 4.0142, in the middle term of the design equation (1.7).

From Figure 1.6, we see that the robust filter (solid) is not at all as sensitive as the nominal filter (dashed) to deviations in the transducer parameter ρ. In fact, the performance is rather close to the lower obtainable limit (dotted). The values of the averaged criterion (1.5) are $J = 1.26$ for the nominal design and $J = 0.63$ for the robust design. If we compare to the "trivial" estimator, using $\hat{u}(k|k) \equiv 0$ (dash-dotted), we see that the robust filter results in a better performance for all $\rho > -0.17$. This is a significant improvement as compared with the nominal design. In Figure 1.6, the price paid for obtaining robustness is evident; the performance for the nominal case is degraded. When considering the improvement in non-nominal situations, this deterioration might be a price well worth paying. □

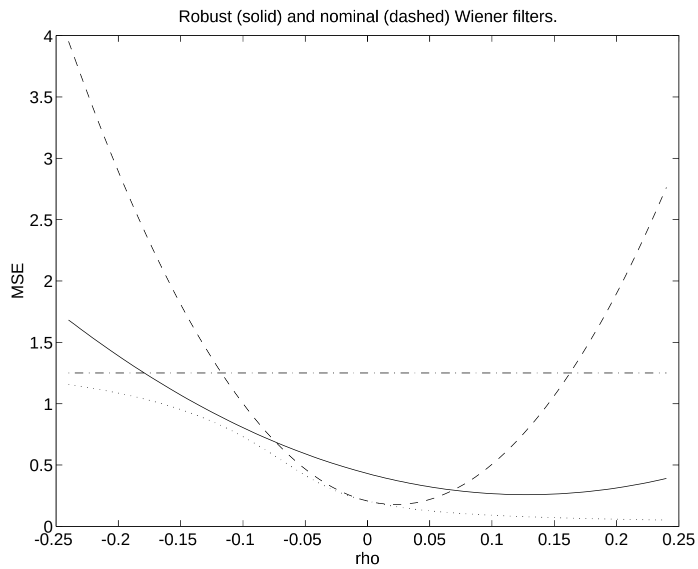

Figure 1.6: This scenario is the same as that presented in Figure 1.5, except that here the performance when the robust Wiener filter (solid) is applied on the set of possible systems is also shown.

1.4 An alternative approach: Uncertainty modelling with hard bounds and minimax $\mathcal{H}_2$-design

There do, of course, exist alternative ways of representing the model uncertainty. A criterion for robust performance is often assumed to be equivalent to *guaranteeing* a pre-specified performance for all systems within the uncertainty set. Such a problem formulation relies on the existence of hard bounds on the uncertainties. Without such bounds, no performance guarantees can be given. Hard bounds in the frequency domain are one commonly used specification in robust filter design. The bounds may then be represented as

$$|\Delta\mathcal{F}(\omega)| \leq L(\omega)$$

where $L(\omega)$ is some frequency-dependent function. In multivariable contexts, bounds on the principal gains

$$\bar{\sigma}\left(\Delta\boldsymbol{\mathcal{F}}(\omega)\right) \leq L(\omega)$$

are frequently used. Here, $\bar{\sigma}(\Delta\boldsymbol{\mathcal{F}}(\omega))$ denotes the largest principal gain of the transfer function matrix $\Delta\boldsymbol{\mathcal{F}}(\omega)$, i.e. the largest singular value as a function of frequency[5]. These and other norm-bounded representations of the spectral uncertainty may be found in, for example, the review by Kassam and Poor [59] or the books by Maciejowski [73] and Boyd and Barrat [16].

As a representation of norm-bounded parametric uncertainties in time-domain models, the state space model

$$\begin{aligned} x(k+1) &= \Big(\mathbf{F}_o + \Delta\mathbf{F}(k)\Big)\, x(k) \\ \Delta\mathbf{F}(k) &= \mathbf{H}\boldsymbol{\Delta}(k)\mathbf{E} \\ \boldsymbol{\Delta}^T(k)\boldsymbol{\Delta}(k) &\leq \mathbf{I}, \qquad k \;=\; 0,1,2,\ldots \end{aligned} \tag{1.8}$$

has recently become popular, see, for example de Souza *et al.* [27], Shaked and de Souza [92], Theodor and Shaked [109] and Bolzern *et al.* [14], [15]. Above, $\mathbf{H}$ and $\mathbf{E}$ are known constant matrices which reflect how the uncertainties affect the nominal state transition matrix $\mathbf{F}_o$.[6]

If performance guarantees are to be provided, it becomes natural to consider some type of minimax optimization. By a minimax approach, we mean minimization problems of the type

$$\min_{\mathcal{R}} \sup_{\Delta, Q} J(\Delta, Q)$$

where Δ and Q represent uncertainties in the signal models and the noise covariances, respectively, while $J(\cdot)$ is some scalar loss-function, for example, the mean square (estimation) error, and $\boldsymbol{\mathcal{R}}$ is the filter sought. Early work on the subject is, for example, described in papers by D'Appolito and Hutchinson [24] and Leondes and Pearson [68]. These papers dealt with large, but bounded, uncertainties in the plant and noise covariances. A paper by Martin and Mintz [76] accounted for both spectral uncertainty and uncertainty in the noise distribution. The resulting robust filters in [76] will, however, tend to be of very high order.

The minimax design of a filter $\boldsymbol{\mathcal{R}}$ becomes very complex, unless there exists either a saddle point or a boundary point solution. A crucial condition here is that

$$\min_{\mathcal{R}} \max_{\mathcal{F}} J(\cdot) = \max_{\mathcal{F}} \min_{\mathcal{R}} J(\cdot)\ . \tag{1.9}$$

If relation (1.9) holds, then one can search the optimal filter which gives the worst nominal performance and then use that filter. As compared with

[5]The principal gain is related to the $\mathcal{H}_\infty$-norm of a transfer function, since $\|\Delta\boldsymbol{\mathcal{F}}(\omega)\|_\infty = \sup_\omega \bar{\sigma}\left(\Delta\boldsymbol{\mathcal{F}}(\omega)\right)$. In the scalar case, $\|\Delta\boldsymbol{\mathcal{F}}(\omega)\|_\infty$ is the peak-value of $|\Delta\boldsymbol{\mathcal{F}}(\omega)|$.

[6]In some formulations also $\mathbf{F}_o$ is allowed to be time-varying.

the task of finding the worst case with respect to a set of models, $\mathcal{F}$, and minimizing with respect to the filter class, $\mathcal{R}$, the former is a much simpler task. However, even when the condition (1.9) is fulfilled, a minimax design can still be computationally demanding. The systematic use of saddle point conditions has been the focus of the work by Kassam *et al.* [58], [80], Poor [90], Vastola and Poor [110] and was reviewed by Kassam and Poor [59]. Haddad and Bernstein [42] propose an alternative approach in which the uncertainties are modelled by white noise sequences.

A problem encountered in minimax designs is that condition (1.9) is *not* fulfilled in numerous problems, which makes them very difficult to solve. See, for instance, Example 5 in [106].

Kalman filter-like estimators have recently been developed for systems with structured and possibly time-varying parametric uncertainty of the type

$$x(k+1) = \Big(\mathbf{F}_o + \mathbf{H}\boldsymbol{\Delta}(k)\mathbf{E}\Big)\, x(k) + w(k)$$

where the matrix $\boldsymbol{\Delta}(k)$ contains norm-bounded, uncertain parameters, see also (1.8) above. This approach originates from work of Ian Petersen and others on the subject of quadratic stabilization. Papers by Shaked and de Souza [92], Petersen and McFarlane [89], Bolzern *et al.* [14] and Xie *et al.* [113] describe continuous-time results. In [114], Xie, Soh and de Souza present a discrete-time, one-step-ahead predictor. This last paper will be considered in more detail in Chapter 7. For systems that are stable for all $\boldsymbol{\Delta}(k)$, an upper bound on the estimation error covariance matrix can be minimized by solving two coupled Riccati-type equations, combined with a one-dimensional numerical search. This represents a computational simplification, as compared with previous minimax designs. The resulting estimators do, however, turn out to be rather conservative, partly because they are based on worst-case design and possibly also because the uncertainty description allows for arbitrary time-variations. See Chapter 7 for a further discussion of this issue.

The method for designing robust filters suggested in the present thesis is computationally less demanding than any of the minimax schemes referred to above. It also avoids two drawbacks of worst-case designs: First, the stochastic variables in the error model $\Delta\mathcal{F}$ need not have compact support. Thus, the descriptions of model uncertainties may have *"soft bounds"*, which are more readily obtainable in a noisy environment than the hard bounds required for minimax design. Second, not only the range of the uncertainties, but also their likelihood is taken into account by taking the expectation $\bar{E}(\cdot)$ of the MSE. Highly probable model errors will affect the estimator design more than very rare "worst cases" will. Therefore, the performance loss in the nominal case, which is a price paid for robustness, becomes smaller than for a minimax design. In other words, conservativeness is reduced. There do, of course, exist applications where a worst-case design is mandatory, e.g. for safety reasons. However, we believe that the average performance of filters is often a more appropriate measure of performance robustness.

Let us consider Example 1.1 once more in order to see what performance we obtain by using a filter designed by minimizing the worst-case MSE.

Example 1.1 *continued. A minimax filter design*

The minimax filter $\mathcal{R}_{minimax}$ was found by making a numerical search within the assumed interval $\rho \in (-0.24, 0.24)$, as the solution to the problem

$$\min_{\mathcal{R}} \sup_{\rho} E(\varepsilon^2(k)) \ .$$

In this example the saddle point condition (1.9) is fulfilled, and the worst attainable performance (the highest value of the dotted curves in Figures 1.5–1.7) is found at the boundary point $\rho = -0.24$. The selected filter, thus, equals the Wiener filter designed for $\rho = -0.24$. The numerical expression of the resulting filter is

$$\mathcal{R}_{minimax}(q^{-1}) \quad = \quad \frac{0.2661}{1 - 1.2027\, q^{-1} + 0.3460\, q^{-2}} \ ,$$

and the performance obtained when $\mathcal{R}_{minimax}$ is applied to different possible systems is displayed in Figure 1.7.

From Figure 1.7, we see that at the boundary point, $\rho = -0.24$, where $\mathcal{R}_{minimax}$ (bold dotted curve) results in the highest MSE, it also, by design, coincides with the optimal Wiener filter (non-bold dotted curve) at that point. Therefore, no other linear estimator can decrease the corresponding MSE value, and we have thus found an optimal minimax filter for the present problem. Also, from Figure 1.7, we see that the minimax filter fulfills its design task, namely, minimizing the maximum value of the MSE within the assumed uncertainty set. But we also see the drawback of the minimax design: The performance for nominal or almost nominal situations is significantly degraded. Figure 1.7 also reveals that the robust Wiener filter, based on the averaged performance criterion, clearly outperforms the minimax filter for most values of ρ. This is true, in particular, for nominal or almost nominal situations where the conservativeness of the

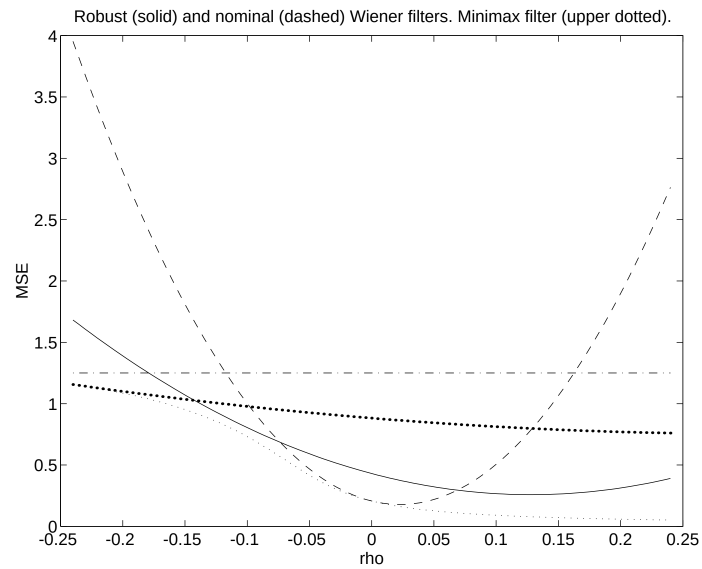

Figure 1.7: The same scenario as in Figure 1.6, but here we also depict the performance when a filter, designed based on a minimax criterion (bold dotted curve), is applied to the set of possible systems. The minimax solution here is a Wiener filter, designed for the boundary point $\rho = -0.24$.

minimax filter is pronounced. In the present problem, the robust Wiener filter performs better or much better over 80% of the range of ρ.[7] □

1.5 Outline of the thesis

The different chapters can be read separately, but we recommend the reader to start with Section 2.2, to get acquainted with the properties of the design model used for the robust designs. We also recommend that the reader, at least, browse through Chapter 4 before reading Chapter 5. The problems presented in these two chapters are dual, and the discussion given in Chapter 5 is not as extensive as the discussion given in Chapter 4. Readers with a particular interest in the design issues can defer reading Chapter 3 until needed. Below an outline of each chapter to follow is presented.

[7]In the remaining range of variation, $-0.24 \leq \rho < -0.14$, the estimation problem is hopelessly difficult: No filtering estimator can perform significantly better than the trivial zero estimator, $\hat{u}(k|k) = 0$ (dash-dotted curve). While the performance of the minimax estimator is better than that of the averaged estimator for $\rho < -0.16$, the resulting performance can hardly be considered useful.

Chapter 2: Probabilistic Error Models. The extended design model is introduced and discussed. Section 2.2 presents the properties and the parameterization of the extended design model. The idea behind the extended design model and the subsequent filter design is illustrated by Figure 2.1. Covariance matrices for the error models and the calculation of averaged polynomial matrices are the subjects of Section 2.3–2.4. Examples are also presented for illustration.

Chapter 3: Obtaining Error Models. In this chapter we discuss both how to obtain error models and how to transform them so as to fit into the extended design model framework. The most simple and pragmatic way of obtaining the extended error models is to tune the error models directly based on *a priori* information. The error models are then used as robustness "tuning knobs". Pragmatic tuning is considered in Section 3.2. Ways in which error models can be obtained from system identification, and from frequency domain data are discussed in Sections 3.3 and 3.6, respectively. The transformation of error models, obtained from first principles, into discrete-time extended design models is the subject of Section 3.4.

The extended design model may also be used to capture slow time-variations in the dynamics of systems, see Section 3.5. In the error modelling, it may happen that the obtained design model is non-linear in the stochastic parameters parameterizing the error model. This can be the case when an ARMA (Autoregressive moving average) model is obtained by means of system identification. Therefore, in Sections 3.7–3.8, we show how extended design models are obtained by means of series expansions when uncertain parameters affect design models in a non-linear way. Both transfer function models (Section 3.7) and state space models (Section 3.8) are considered. Several examples are provided to illustrate the results. The chapter is concluded with some remarks in Section 3.9.

Chapter 4: Robust Multivariable $\mathcal{H}_2$ Estimation. This chapter, which constitutes the central part of the thesis, concerns the multivariable Wiener estimator for smoothing, prediction, filtering and deconvolution. The problem formulation is introduced in Section 4.2, see Figure 4.1, followed by a general parameterization of the design models in Section 4.3. Based on these two sections, the derivation of a robust multivariable Wiener filter is presented in Section 4.4. The resulting filter is a generalization of the estimator for scalar problems presented by Sternad and Ahlén [106]. It is also a generalization to robust design of the nominal multivariable Wiener filter, presented in [2], [3] by Ahlén and Sternad.

The general solution of Section 4.4 includes a rather large number of design equations in a multivariable setting. A radical simplification is both desirable and possible. For this purpose, a simplified parameterization of the models is presented in Section 4.5. The use of matrix fraction descriptions with diagonal denominators and common denominator forms leads to a solution which, in fact, is significantly simpler and more numerically well-behaved than the corresponding nominal $\mathcal{H}_2$-designs (without uncertainty) presented in [2] or [39]. It turns out that taking model uncertainty into account in the filter design does not require any new types of design equations. We end up with two equations for robust estimator design: a polynomial matrix spectral factorization and a unilateral polynomial matrix Diophantine equation. (In the general solution of Section 4.4, coprime factorizations of polynomial matrices also have to be performed.) The solution provides structural insight; important properties of a robust estimator are evident by direct inspection of the filter expression. The special case of estimating signals in white noise is considered in Section 4.6. In this case an explicit solution of the Diophantine equation can be found[8]. Thus, only one design equation, namely the spectral factorization, has to be solved when estimating signals in white noise.

Analytical expressions for evaluating the performance of the robust filter are given in Section 4.7. Since the extended design models are to be linear in the uncertain parameters, it is, mathematically, possible to obtain a filter identical to the robust filter by assuming a suitable, artificial, noise statistics in a conventional Wiener filter design. This is the subject of Section 4.8. From this interpretation we learn that model uncertainty in the input signal model has the same effect as an increase in the signal-to-noise ratio, while uncertainties in the disturbance models are equivalent to a decrease in the signal-to-noise ratio. Uncertainties in the transducer models act like additive measurement noise. The impact of these uncertainties depend on the energy of the input signals in the corresponding frequency regions. We also discuss the feasibility of directly robustifying a (nominal) Wiener filter design by representing the model uncertainties as additive coloured noises. The chapter is concluded by presenting a detailed step-by-step design example in Section 4.9.

Chapter 5: Robust Multivariable $\mathcal{H}_2$ Feedforward Control. This chapter deals with the design of multivariable, robust feedforward controllers. Because of the dual relation between feedforward control and deconvolution, demonstrated by Bernhardsson and Sternad in [11], this chap-

[8] This property first noted by Lindbom in [70].

ter is not as comprehensive as Chapter 4. Section 5.2 presents the problem formulation. Thereafter, in Section 5.3–5.4, the multivariable robust feedforward controller is derived for a general setup, followed in Section 5.5 by a simplified robust design in the same spirit as in Section 4.5. The results of Chapter 5 are a generalization to multivariable systems of the scalar robust feedforward controller presented by Sternad and Ahlén in [106]. The proposed design also generalizes the multivariable feedforward controller presented by Sternad and Ahlén in [105], [104] to the case with uncertain models.

Both Chapter 4 and Chapter 5 constitute a generalization to uncertain design models of the polynomial equations methodology, pioneered in [62], and further developed in [63], by Kučera.

Chapter 6: Robust State Estimation. Using a state space formulation, the stochastic extended design models and the averaged MSE criterion, robust state estimators are derived in this chapter. One of the aims of Chapter 6 is to illustrate how the probabilistic error models may be incorporated to robustify, not only Wiener estimators, but also Kalman-like state estimators and adaptation algorithms for parameter tracking. It is shown how the extended design models can be used to obtain robust Kalman state estimators for prediction, filtering and fixed lag smoothing in Section 6.2. The robustness is obtained by the means of extending the state space models and re-defining the noise covariance matrices to account for the assumed model uncertainty. This is in perfect agreement with the discussion given in Section 4.8. An example illustrating the design of robust state estimators is given at the end of Section 6.2.

Using the results of the Section 6.2, algorithms can be formulated for adaptive estimation of time-varying parameters in linear regressor models, based on uncertain *a priori* information on the statistical properties of these parameter variations. A robust tracking algorithm, which complements the Wiener-based methodology for designing tracking algorithms developed by Lindbom in [70], is derived in Section 6.3. The algorithm is illustrated in Chapter 7.

Chapter 7: A Concluding Example. In this chapter a detailed comparison with a minimax $\mathcal{H}_2$ approach to robust filtering is presented. We consider an example presented by Xie *et al.* in [115] and further examined by Theodor and Shaked in [109]. The problem formulation is given in Section 7.3. The approach proposed in [115] is reviewed in Section 7.3, while the results presented in [109] are given in Section 7.4. It is shown that the approach proposed in [115] results in a very conservative design, while the design as proposed in [109] performs better. As a comparison, we use the results from Section 3.7 to obtain a suitable extended design model in Section 7.5. Using this model a robust Wiener filter, according to the results of Chapter 4, is designed in Section 7.6. Thereafter, in Section 7.7, a robust Kalman predictor is derived along the lines of Chapter 6. Comparisons to the estimator proposed in [115] and [109] are summarized by a discussion in Section 7.8.

Chapter 8: Conclusions. Here the thesis is concluded by presenting a summary of the results obtained. In addition, suggestions for future research are offered and discussed.

Appendix A: Solving the Bilateral Diophantine Equation. This appendix contains an algorithm for solving the bilateral polynomial matrix Diophantine equation encountered in Section 4.4 and 5.4.

Appendix B: MATLAB™ Algorithms. Here, a short description of some MATLAB™ .m files for working with the robust Wiener filter design using MATLAB™ is included. These .m files can be obtained upon request.